FREE VIBRATION ANALYSIS OF BEAM ELEMENT USING ISOGEOMETRIC ANALYSIS

Miloš Jočković [1]
Marija Nefovska Danilović [2]
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Summary: *Isogeometric analysis (IGA) is based on a concept that uses the same base functions for description of displacement field and undeformed model geometry. The most common base functions used in the IGA are NURBS functions. In this paper, the IGA is applied in the free vibration analysis of beam element. The stiffness and mass matrices have been developed for rotation-free Bernoulli-Euler and Timoshenko beam using the Galerkin method. Natural frequencies of beam element with specific boundary conditions have been computed using the isogeometric approach. The results were compared with the exact analytical solutions obtained by using the dynamic stiffness method (DSM) and the conventional finite element method (FEM).*

Keywords: *isogeometric analysis, free vibrations, Bernoulli-Euler beam, Timoshenko beam*

1. INTRODUCTION

Beam elements represent one of the basic components in civil engineering structures. These structures are often subjected to dynamic excitation and thus the free vibration analysis of beams is of great importance. The vibration analysis is usually conducted using Bernoulli-Euler beam, but for thick beams, especially for high modes of vibration this theory does not provide adequate results. In such cases, the Timoshenko beam has to be used where the transverse shear deformation and rotatory inertia effect has been included [1]. Different approaches can be used for free vibration analysis of engineering structures such as finite element method (*FEM*), dynamic stiffness method (*DSM*), isogeometric analysis (*IGA*), etc.
The IGA was first introduced by Hughes [2]. Hughes and his associates presented capability and potential for usage of NURBS basic functions in structural analysis. Previously, NURBS functions, developed by Piegl [3], were used for representation, design, and data exchange of geometric information processed by computers. Cottrell et al. [4] used the IGA in vibration analysis of elastic rod, Bernoulli-Euler beam, membrane

[1] Miloš Jočković, MSc Civil Eng., University of Belgrade, Faculty of Civil Engineering, Bulevar kralja Aleksandra 73, 11000 Belgrade, Serbia, tel: ++381 11 3218 581, e – mail: mjockovic@gfr.bg.ac.rs
[2] Marija Nefovska Danilović, PhD Civil Eng., University of Belgrade, Faculty of Civil Engineering, Bulevar kralja Aleksandra 73, 11000 Belgrade, Serbia, tel: ++381 11 3218 552, e – mail: marija@grf.bg.ac.rs

and Poisson-Kirchhoff plate. Also, Lee and Park [5] analysed the free vibration of Timoshenko beam for various boundary conditions using the isogeometric approach.
In this paper the isogeometric approach is applied in the free vibration analysis of Bernoulli-Euler and Timoshenko beams. The isogeometric elements have been implemented in the program coded in MATLAB [6] and used to compute free vibration characteristics of straight beam with specific boundary conditions.

2. ISOGEOMETRIC ANALYSIS

Isogeometric analysis is approach based on idea that the analysis model uses the same mathematical description as the geometry model. The approach states that the same functions used for geometry description in CAD (*Computer Aided Design*) are adopted in the analysis for the geometry and solution field. This approach is useful because it merges design and analysis into one model.
In the IGA, the domain consists of couple patches and each patch can be considered as a macro-element. The patch is defined over a parametric domain, which is divided by a knot vector. The intervals defined by a knot vector represent the IGA element. Similar to the FEM, an IGA element is specified by a set of nodes and corresponding basic functions. The nodes are IGA control points. They carry degree of freedom for the analysis and the boundary conditions are applied to them. As mentioned before, the basic functions used in the IGA are the same functions used for geometry description of the model. In general, these functions are not bound to one IGA element but extended over a series of elements. This property of basic functions allows higher continuities of shape functions over the element boundaries. However, these elements can be treated exactly in the same way as the conventional finite elements. The stiffness and mass matrices are evaluated on the element level and assembled to the global stiffness and mass matrices. More about the knot vector and basic functions will be presented in following.

2.1 Knot vector

A knot vector θ represents a set of non-decreasing real values ξ_i, known as knots, in the parametric space:

$$\theta = \left[\xi_1,\ \xi_2,\ \xi_3,\ \ldots,\ \xi_i,\ \ldots,\ \xi_{n+p+1} \right] \quad (1)$$

where n is a number of basic functions and p is an order of the basic function. A knot vector is said to be uniform if its components are uniformly spaced. Moreover, a knot vector is said to be open or clamped if its first and last knots have the multiplicity equals to p+1. Basic function formed from open knot vector has interpolatory property at the ends of the parametric interval but has not, in general, the interpolatory property at the interior knots. In this paper the open knot vector is considered.

2.2 B-spline basic functions

B-spline basic functions are defined recursively starting with p = 0 as:

$$N_{i,0}(\xi)=\begin{cases}1 & \text{if } \xi_i \le \xi \le \xi_{i+1}\\ 0 & \text{otherwise,}\end{cases} \tag{2}$$

and

$$N_{i,p}(\xi)=\frac{\xi-\xi_i}{\xi_{i+p}-\xi_i}N_{i,p-1}(\xi)+\frac{\xi_{i+p+1}-\xi}{\xi_{i+p+1}-\xi_{i+1}}N_{i+1,p-1}(\xi) \tag{3}$$

for p>0.

The functions defined in Eq. (2) and Eq. (3) fulfil the necessary conditions for basic functions, such as linear independence and partition of unity. Also, the non-negativity of the functions affects the mass matrix property of the isogeometric element i.e. all terms in this matrix are positive valued terms. Also, if the interval knots are not repeated, the functions are C^{p-1} continuous. However, if a knot has the multiplicity *k*, the function is C^{p-k} continuous at the particular knot. This means that basic function may have interpolatory property at the interior knot if the knot has the multiplicity *p*.

2.3 B-spline curves

B-spline curve $S(\xi)$ of order *p* can be obtained as a linear combination of B-spline basic functions:

$$S(\xi)=\sum_{i=1}^{n}N_{i,p}(\xi)C_i \tag{4}$$

where *n* is the number of control points, $N_{i,p}(\xi)$ are B-spline basic functions of order *p* and C_i is control point. It should be noted that the B-spline curves also have the same property as the basic functions defined in the previous section.

2.4 NURBS

NURBS is rational representation of the B-spline curve. The p^{th}-degree NURBS curve is defined analogously to (4) as:

$$S(\xi)=\frac{\sum_{i=1}^{n}N_{i,p}(\xi)\,f_iC_i}{\sum_{i=1}^{n}N_{i,p}(\xi)\,f_i} \tag{5}$$

where *n* is the number of control points, $N_{i,p}(\xi)$ is a B-spline basic function of the order *p*, C_i is control point and f_i is weight. Curves defined by Eq. (5) can be used to exactly model the geometry of circles, ellipses, hyperbolas, etc. that cannot be modelled with the B-spline curves. More about these functions, their properties and application can be found in literature [3].

3. ISOGEOMETRIC FORMULATION

In this chapter the isogeometric approach will be used to solve the free vibration problem of beam element. Two beam theories, Bernoulli-Euler and Timoshenko beam theory will be considered.

3.1 Bernoulli-Euler beam theory

The governing differential equation of motion for Bernoulli-Euler beam is given as:

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}=0 \tag{6}$$

where *w=w(x,t)* is the transverse displacement, *E* is the elastic modulus, *I* is the second moment of area, ρ is the mass density of material, *A* is the cross sectional area.
Using the Galerkin method, a weak formulation of the free vibration problem is obtained as:

$$\int_0^L EI\frac{\partial^2 w}{\partial x^2}\delta\left(\frac{\partial^2 w}{\partial x^2}\right)+\int_0^L \rho A\frac{\partial^2 w}{\partial t^2}\delta w=0 \tag{7}$$

where *L* is the length of the beam and the notation δ denotes that the term is virtual.
In order to transform the Eq. (6) into a system of algebraic equations, the relevant derivation takes place in the finite-dimensional subspace. Since only straight beam will be analyzed in this paper, the weights in Eq. (5) are equal to 1. Thus, the subspaces are defined by using the B-spline basis:

$$w=\sum_{i=1}^{m}N_i\left(\xi\right)\hat{w}_i=\mathbf{N}\hat{\mathbf{w}} \tag{8}$$

where *m* is the total number of control points in the discretized domain, **N** is the vector consisting of basic functions, while $\hat{\mathbf{w}}$ represents the vector consisting of control points. The virtual term associated with the displacements is:

$$\delta w=\sum_{i=1}^{m}N_i\left(\xi\right)\delta\hat{w}_i=\mathbf{N}\delta\hat{\mathbf{w}} \tag{9}$$

By substituting Eq. (8) and Eq. (9) into Eq. (7) yields:

$$\delta\hat{\mathbf{w}}\left[\mathbf{K}\hat{\mathbf{w}}+\mathbf{M}\frac{\partial^2\hat{\mathbf{w}}}{\partial t^2}\right]=0 \tag{10}$$

Since the virtual displacement $\delta\hat{\mathbf{w}}$ is arbitrary, the above equation may be written as:

$$\mathbf{K}\hat{\mathbf{w}} + \mathbf{M}\frac{\partial^2 \hat{\mathbf{w}}}{\partial t^2} = 0 \tag{11}$$

In Eq. (10) and Eq. (11) **K** and **M** represents respectively the structural stiffness and mass matrices and can be explicitly written as:

$$\mathbf{K} = \int_0^L \frac{\partial^2 \mathbf{N}}{\partial x^2} EI \frac{\partial^2 \mathbf{N}^T}{\partial x^2} dx \qquad \mathbf{M} = \int_0^L \mathbf{N}\rho A \mathbf{N}^T dx \tag{12}$$

General solution of Eq. (11) may be written as:

$$\hat{\mathbf{w}} = \boldsymbol{\psi}_k \mathrm{e}^{i\omega_k} \tag{13}$$

Substituting Eq. (13) into Eq. (11) yields:

$$\left[\mathbf{K} - \omega_k^2 \mathbf{M}\right]\boldsymbol{\psi}_k = 0 \tag{14}$$

where $\boldsymbol{\psi}_k$ is the modal vector, ω_k is the natural circular frequency associated with the k^{th} mode. Global stiffness matrix **K** and global mass matrix **M** contain the contributions of the isogeometric element stiffness and mass matrices.
As mentioned before, the basic functions at the interior knots don't have the interpolatory property and the rotation cannot be imposed directly. This is the main reason for introducing the rotation-free beam element. The main property of this element is that it has no rotation degrees of freedom i.e. the displacements are the only considered degrees of freedom. The problem related to the rotation-free element arises when the rotation boundary condition has to be enforced. In order to solve this problem Lagrange multiplier can be used [4]. More about this method will be presented in the numerical example.

3.2 Timoshenko beam

The governing differential equations of motion for Timoshenko beam are given as:

$$kAG\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \varphi}{\partial x}\right) - \rho A \frac{\partial^2 w}{\partial t^2} = 0 \qquad EI\frac{\partial^2 \varphi}{\partial x^2} + kAG\left(\frac{\partial w}{\partial x} - \varphi\right) - \rho I \frac{\partial^2 \rho}{\partial t^2} = 0 \tag{16}$$

where w is transverse displacement, φ is rotation of the beam, k is the shear coefficient, A is the cross section area, G is the shear modulus, ρ is the mass density, E is the elastic modulus, I is the second moment of area.

The weak formulation of Eq. (16) is obtained by using the Galerkin method in the following form:

$$\int_0^L \underbrace{\begin{bmatrix} w & \varphi \end{bmatrix}}_{\mathbf{q}^T} \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ -1 & \frac{\partial}{\partial x} \end{bmatrix}}_{\mathbf{L}^T} \underbrace{\begin{bmatrix} kAG & 0 \\ 0 & EI \end{bmatrix}}_{\mathbf{E}_K} \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & -1 \\ 0 & \frac{\partial}{\partial x} \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \delta w \\ \delta\varphi \end{bmatrix}}_{\delta\mathbf{q}} dx + \int_0^L \frac{\partial^2}{\partial t^2} \underbrace{\begin{bmatrix} w & \varphi \end{bmatrix}}_{\mathbf{q}^T} \underbrace{\begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix}}_{\mathbf{E}_M} \underbrace{\begin{bmatrix} \delta w \\ \delta\varphi \end{bmatrix}}_{\delta\mathbf{q}} dx = 0 \tag{17}$$

where L is the length of the beam element, $\mathbf{q}$ is the vector of generalised displacements, $\mathbf{L}$ is operator matrix, $\mathbf{E}_K$ is the matrix consisting of geometric and material properties of the beam, $\mathbf{E}_M$ is the matrix consisting of inertia properties of beam and δ denotes virtual generalised displacements.
The subspaces for Timoshenko beam isogeometric element are defined as:

$$\begin{gathered} w = \sum_{i=1}^{m} N_i(\xi)\hat{w}_i \qquad \varphi = \sum_{j=1}^{m} N_j(\xi)\hat{\varphi}_j \\ \mathbf{q} = \begin{bmatrix} w \\ \varphi \end{bmatrix} = \mathbf{N}\hat{\mathbf{q}} \\ \mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_m & 0 \\ 0 & N_1 & 0 & N_2 & \cdots & 0 & N_m \end{bmatrix} \\ \hat{\mathbf{q}}^T = \begin{bmatrix} \hat{w}_1 & \hat{\varphi}_1 & \hat{w}_2 & \hat{\varphi}_2 & \cdots & \hat{w}_m & \hat{\varphi}_m \end{bmatrix} \end{gathered} \tag{18}$$

The virtual terms associated with the displacements are:

$$\begin{gathered} \delta w = \sum_{i=1}^{m} N_i(\xi)\delta\hat{w}_i \qquad \delta\varphi = \sum_{j=1}^{m} N_j(\xi)\delta\hat{\varphi}_j \\ \delta\mathbf{q} = \begin{bmatrix} \delta w \\ \delta\varphi \end{bmatrix} = \mathbf{N}\delta\hat{\mathbf{q}} \\ \mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_m & 0 \\ 0 & N_1 & 0 & N_2 & \cdots & 0 & N_m \end{bmatrix} \\ \delta\hat{\mathbf{q}}^T = \begin{bmatrix} \delta\hat{w}_1 & \delta\hat{\varphi}_1 & \delta\hat{w}_2 & \delta\hat{\varphi}_2 & \cdots & \delta\hat{w}_m & \delta\hat{\varphi}_m \end{bmatrix} \end{gathered} \tag{19}$$

By substituting Eq. (18) and Eq. (19) into Eq. (17) Eq. (14) is obtained and the structural stiffness and mass matrices are given as:

$$\mathbf{K} = \int_0^L \mathbf{N}^T \mathbf{L}^T \mathbf{E}_K \mathbf{L} \mathbf{N} dx$$
$$\mathbf{M} = \int_0^L \mathbf{N}^T \mathbf{E}_M \mathbf{N} dx \qquad (20)$$

4. NUMERICAL EXAMPLE

The isogeometric approach presented in the previous sections is applied in the free vibration analysis of straight beam given in Figure 1. The Poisson's ratio is ν = 0.2 and the shear coefficient is k = 5/6.

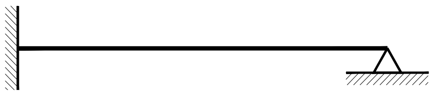

Figure 1: Beam element and boundary conditions

In order to illustrate the convergence and accuracy of the isogeometric approach, the first 10 dimensionless natural frequencies $\Omega_i = \frac{\omega_i}{2\pi} L^2 \sqrt{\frac{\rho A}{EI}}$ are computed. The results are compared with the results obtained by using the dynamic stiffness method (DSM) [7] and the finite element software Abaqus [8]. The DSM is based on the exact solutions of the equations of motion and consequently, yields the exact natural frequencies for beam element. In the numerical examples for IGA the basic functions of the second order are applied. The number of the isogeometric elements has been changed in order to analyse the convergence and accuracy of the results.

4.1 Bernoulli-Euler beam

The Bernoulli-Euler beam is first investigated by the presented isogeometric element. As illustrated in Figure 1, the left end of the beam is clamped i.e. the displacement and rotation is zero. In order to impose this boundary condition for rotation-free beam the Lagrange multiplier is used and Eq. (7) is modified by introducing the additional term:

$$\int_0^L EI \frac{\partial^2 w}{\partial x^2} \delta\left(\frac{\partial^2 w}{\partial x^2}\right) + \delta\lambda \left.\frac{\partial w}{\partial x}\right|_{x=0} + \lambda\delta \left.\frac{\partial w}{\partial x}\right|_{x=0} + \int_0^L \rho A \frac{\partial^2 w}{\partial t^2} \delta w = 0$$
$$\left.\frac{\partial w}{\partial x}\right|_{x=0} = 0 \qquad (21)$$

where λ is Lagrange multiplier.

The dimensionless natural frequencies are computed for different number of isogeometric elements and presented in Table 1. Moreover, in this table the dimensionless natural frequencies have also been computed by the DSM and Abaqus. The beam is modelled using 500 finite elements using Abaqus. It can be noted that as the number of isogeometric elements increases the natural frequencies computed using the IGA converge to the exact solutions.

Table 1: First ten dimensionless frequencies of Bernoulli-Euler beam

Mode No.	IGA					Exact solution (DSM)	Abaqus
	10	20	30	40	50		
1	2.48	2.46	2.46	2.46	2.45	2.46	2.45
2	8.20	8.01	7.98	7.97	7.96	7.96	7.95
3	17.58	16.83	16.70	16.65	16.63	16.59	16.59
4	31.19	29.03	28.66	28.53	28.48	28.38	28.37
5	49.85	44.79	43.95	43.66	43.53	43.34	43.29
6	74.58	64.32	62.65	62.08	61.82	61.36	61.36
7	106.04	87.89	84.86	83.84	83.38	82.59	82.57
8	141.88	114.84	110.73	109.03	108.26	106.9	106.91
9	170.0	148.54	140.41	137.73	136.52	134.39	134.4
10	-	186.45	174.08	170.02	168.20	165.05	165.03

The first five mode shapes obtained using the isogeometric approach are presented in Figure 2.

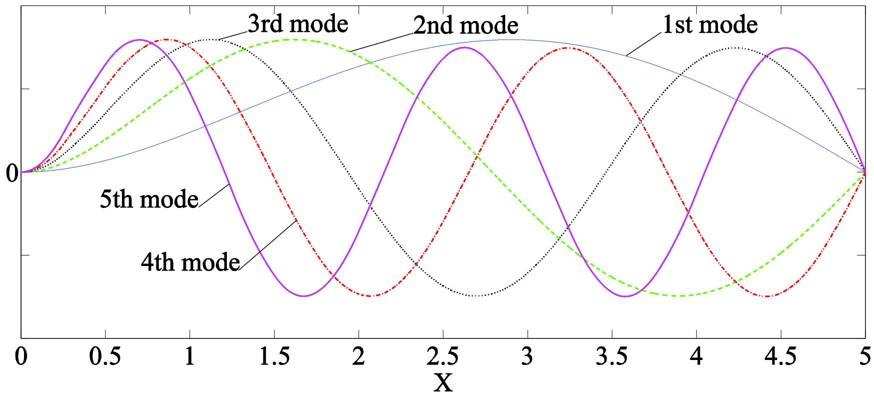

Figure 2: First five mode shapes of Bernoulli-Euler beam

4.2 Timoshenko beam

The dimensionless natural frequencies of Timoshenko beam computed using the isogeometric approach are compared with the results obtained using the DSM and Abaqus. As in previous example, the beam is modelled using 500 finite elements. The results are presented in Table 2. Excellent agreement has been achieved between the present solution and the solutions obtained using the DSM and Abaqus for lower vibration modes. For higher vibration modes, a larger number of isogeometric elements are required in the analysis. The first five mode shapes of Timoshenko beam are presented in Figure 3.

Table 2: First ten dimensionless natural frequencies of Timoshenko beam

Mode No.	IGA					Exact solution (DSM)	Abaqus
	10	20	30	40	50		
1	2.33	2.33	2.33	2.33	2.33	2.33	2.33
2	7.02	7.00	7.00	7.00	7.00	7.00	7.02
3	13.39	13.33	13.33	13.33	13.33	13.26	13.37
4	20.93	20.70	20.70	20.69	20.69	20.43	20.77
5	29.43	28.73	28.70	28.70	28.70	28.06	28.82
6	39.02	37.16	37.11	37.10	37.10	35.90	37.28
7	50.13	45.88	45.76	45.75	45.74	43.83	45.99
8	63.00	54.80	54.58	54.54	54.54	51.75	54.86
9	75.91	63.91	63.48	63.43	63.41	59.66	63.81
10	78.62	73.18	72.44	72.34	72.32	67.56	72.80

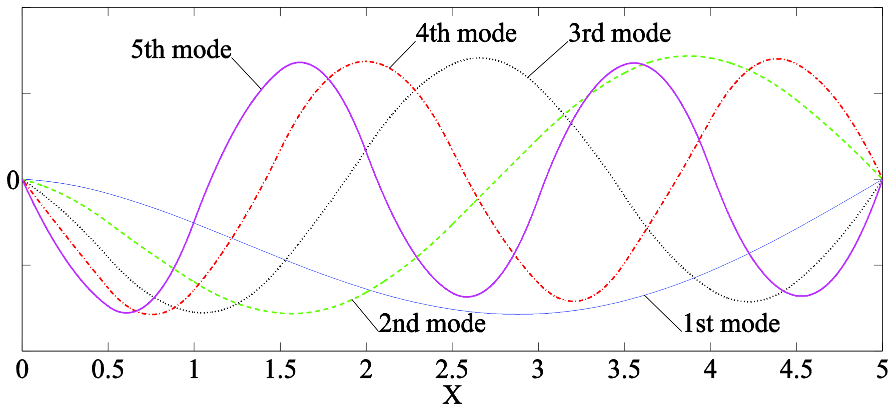

Figure 3: First five mode shapes of Timoshenko beam

5. CONCLUSION

The application of the isogeometric approach in the free vibration analysis of beam element is presented in this paper. The B-spline basic functions have been used for general displacement representation. The stiffness and mass matrices have been developed for Bernoulli-Euler and Timoshenko beam theory. In the case of Bernoulli-Euler beam theory the rotation-free element is introduced by using the Lagrange multiplier in order to enforce the rotation boundary condition. The free vibration analysis has been conducted for beam with specific boundary conditions using the isogeometric approach. The numerical example has shown good performance and accuracy of the method in comparisson with the results obtained from the DSM and FEM.

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REFERENCES

[1] Timoshenko, S.P.: On the correction for shear of the differential equation for transverse vibrations of prismatic bars. *Philosophical Magazine*, **1921**, vol. 41, p.p. 744-746
[2] Cottrell, J.A., Hughes, T.J.R., Bazilevs, Y.: *Isogeometric Analysis: Toward Integration of CAD and FEM*, Wiley, UK, **2009**.
[3] Piegl, L., Tiller, W.: *The NURBS Book*, Springer, USA, **1995**.
[4] Cottrell, J.A., Reali, A., Bazilevs, Y., Hughes, T.J.R.: Isogeometric analysis of structural vibrations. *Computer methods in applied mechanics and engineering*, **2006**, vol. 195, p.p. 5257-5296.
[5] Lee, S.J., Park, K.S.: Vibration of Timoshenko beams with isogeometric approach. *Applied Mathematical Modeling*, **2013**, vol. 37, p.p. 9174-9190
[6] *The Language of Technical Computing - MATLAB 2011b*, MathWorks Inc, **2011**.
[7] Šavija, B. Dinamička analiza ramovskih konstrukcija u frekventnom domenu primenom Metode spektralnih elemenata, MSc Thesis, Belgrade, **2009**.
[8] Abaqus. *Abaqus/CEA User's Manual*, Simulia, **2012**.

АНАЛИЗА СЛОБОДНИХ ВИБРАЦИЈА ГРЕДЕ КОРИШЋЕЊЕМ ИЗОГЕОМЕТРИЈСКЕ АНАЛИЗЕ

Резиме: *Изогеометријска анализа (ИГА) се базира на концепту коришћења истих функција за описивање поља померања констукције као и геометрије недеформисаног модела. Најчешће коришћене базне функције у ИГА су NURBS фунцкије. У овом раду, ИГА је коришћена у анализи слободних вибрација гредног елемента. Да би се добили потребни резултати, матрица крутости и матрица маса је изведена за Bernoulli-Euler-ову греду код које су непознате само попречна померања, као и Timoshenko-ву греду помоћу Galerkin-ове методе. Нумеричка анализа приказана је на примеру гредног елемента са конкретним граничним условима. Тачност резултата добијених помоћу ИГА је проверена поређењем са резултатима добијеним методом динамичке крутости и методом коначних елемената.*

Кључне речи: *Изогеометријска анализа, слободне вибрације, Bernoulli-Euler-ова греда, Timoshenko-ва греда*