A NOTE ON THE EQUILIBRIUM ANALYSIS OF DRAPED MASONRY ARCH

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Summary: *Equilibrium analyses of masonry arches usually consider their convex forms, such as circular, elliptical or pointed. However, besides them, there are also different concave or inflectional arches' shapes. Therefore, the aim of this paper is to apply the thrust line theory to a basic arch of the concave form – a draped arch, which consists of two annular sectors leaning against each other, oriented oppositely in comparison to the semicircular arch. In accordance with the applied normal stereotomy, which implies generic sections perpendicular to arch's axis, the problem of equilibrium under self-weight is analytically treated, and the closed-form expression for the thrust line is determined. Moreover, limit equilibrium state is considered, and the analytical solution for the minimum theoretical thickness of draped arch is provided.*

Keywords: *thrust line, masonry arch, draped arch, minimum thickness, static approach, limit equilibrium analysis*

1. INTRODUCTION

Equilibrium analyses of masonry arches usually consider their common convex forms, such as circular [1, 2], elliptical [3] or pointed [4]. However, besides them, there are also different concave, inverted or inflectional arches' shapes (see [5] and few examples shown in Fig. 1a,c,d). Therefore, the aim of this paper is to apply geometric formulation (static approach) in equilibrium analysis to a basic arch of such form – a draped arch. Namely, draped (or tented) arch consists of two annular sectors leaning against each other, oriented oppositely in comparison to the semicircular arch, as one can see in Fig. 1a. Its shape resembles the shape of a tied drapery i.e. long heavy curtain (Fig. 1b), which was a cause for its name.
Analytical modelling of the problem is based on thrust line theory, where thrust line represents the locus of the application points of the resultant thrust forces at the joints between the voussoirs of a structure [1, 2, 3, 4]. Therewith, common assumptions about masonry properties are adopted: no tension strength, infinite compression strength and sliding cannot occur. Thus, the possibility of failure due to material strength or due to sliding is eliminated, permitting only the collapse due to instability, by relative rotation of structure's parts around the edge of the joint of rupture. Accordingly, if a structure is of sufficient thickness, and a thrust line is lying everywhere within the structure's

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boundary (between intrados and extrados), the structure is safe [6]. In order to apply a geometric formulation, i.e. derivation of the limiting thrust line, to a draped arch, its geometrical properties have to be thoroughly treated.

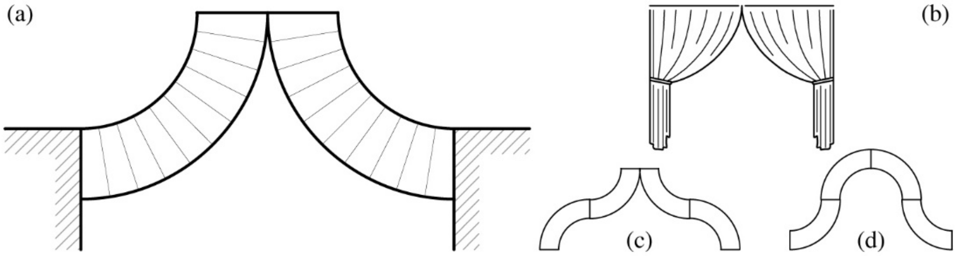

Figure 1. (a) Draped arch, (b) drapery, curtain, (c) ogee arch, (d) bell-shaped arch

2. ANALYTICAL MODELLING

Due to the symmetry of the arch, in the following discussion only a half-arch shown in Fig. 2 is considered. It is the monolithic arch of zero tensile strength and therefore acts only in compression. R and t denote the mean radius and the thickness of the arch ring, respectively. The minimum value of thickness to radius ratio, t/R_{min}, represents the minimum possible thickness of the arch normalized by the radius, and is of particular interest of this paper.

Normal i.e. radial stereotomy, which assumes the directions of the joints between voussoirs concurrent to the centre of an arch i.e. generic sections perpendicular to arch's axis (c. Figs. 1a and 2a,c), is applied. Thus, the forthcoming analytical modelling employs polar coordinates, with the origin set in the half-arch's centre O, as shown in Fig. 2a,c. The angle φ is the angular coordinate, measured from the top horizontal edge (summit) toward the springing, which defines the generic section (see Fig. 2c). Accordingly, the weight V of a finite upper portion of the arch, between the summit and a generic section at the angle φ, is represented by the area of the corresponding arch ring:

$$V = R\,t\,\varphi \tag{1}$$

The abscissa x_V of the centre of gravity of the upper portion of the arch (Fig. 2b), i.e. the centroid of the area which corresponds to the weight V, can be computed according to the following expression:

$$x_V(\varphi) = \frac{2}{3}\frac{\left(R+\frac{t}{2}\right)^3-\left(R-\frac{t}{2}\right)^3}{\left(R+\frac{t}{2}\right)^2-\left(R-\frac{t}{2}\right)^2}\frac{\sin\left(\frac{\varphi}{2}\right)}{\frac{\varphi}{2}}\cos\left(\frac{\varphi}{2}\right) = \frac{(12\,R^2+t^2)\sin\varphi}{12\,R\,\varphi} \tag{2}$$

When the generic angle φ reaches right angle, the weight W of a half-arch shown in Fig. 2a is obtained:

$$W = R\,t\,\frac{\pi}{2} \tag{3}$$

Therewith, the abscissa x_W of the centre of gravity of the half-arch is simplified from Eq. (2) to the following expression:

$$x_W = x_V\left(\varphi = \frac{\pi}{2}\right) = \frac{12\,R^2 + t^2}{6\,R\,\pi} \tag{4}$$

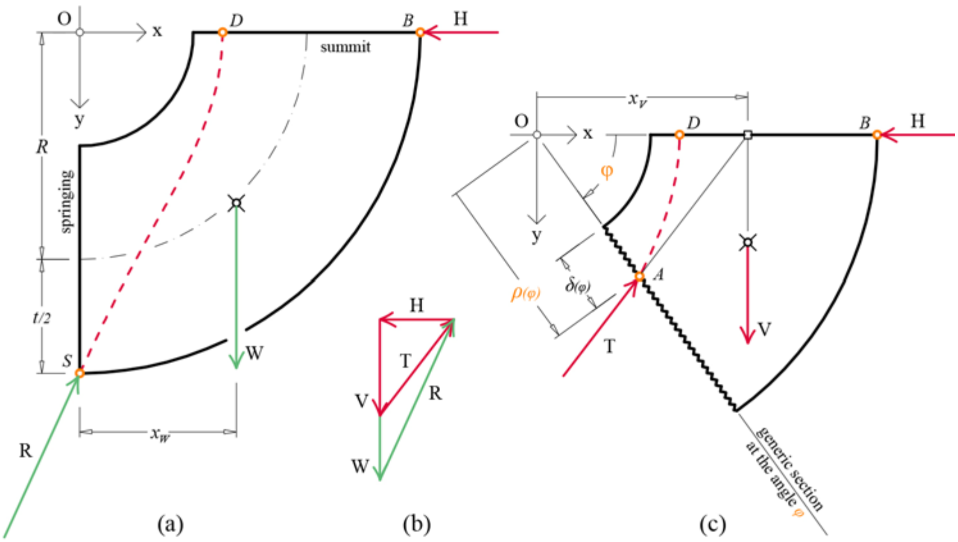

Figure 2. Draped arch: (a) geometric parameters of half-arch, (b) force polygon, (c) free-body diagram of the isolated finite top portion up to the generic section

Since the halves of the draped arch are leaning against each other at the apex in a single point *B*, the horizontal thrust force *H* must pass through that point (see Fig. 1a and Fig. 2a). On the other hand, since the minimum thrust line, having the greatest rise, is assumed, the application point *S* of the reaction force *R* is positioned at the bottom end of the springing. Accordingly, the rotational equilibrium of a half-arch about the point *S* is expressed by the following equality:

$$H\left(R + \frac{t}{2}\right) = W\,x_W \tag{5}$$

whereas the values *W* and x_W are given by Eqs. (3) and (4), respectively. Thus, one can derive the value of horizontal thrust:

$$H = \frac{W\,x_W}{R + \frac{t}{2}} = \frac{12\,R^2\,t + t^3}{12\,R + 6\,t} \tag{6}$$

The resultant thrust force *T* at a generic section at the angle φ together with its point of application *A* is uniquely determined from the force and moment equilibrium of the finite portion of the arch; it can be done either graphically with the force polygon (Fig. 2b,c) or analytically by solving equilibrium equations. Accordingly, from rotational equilibrium about point *A* follows:

$$H\,\rho(\varphi)\sin\varphi = V(\varphi)\left(x_V(\varphi) - \rho(\varphi)\cos\varphi\right) \tag{7}$$

Finally, from the previous equality, one can determine the distance ρ between the thrust line and the centre of the arch, deriving the closed-form expression for the thrust line within draped arch:

$$\rho(\varphi) = \frac{V(\varphi)\,x_V(\varphi)}{H\sin\varphi + V(\varphi)\cos\varphi} = \frac{1}{\dfrac{2}{2\,R+t} + \dfrac{12\,R\,\varphi\cot\varphi}{12\,R^2 + t^2}} \tag{8}$$

where $V(\varphi)$, $x_V(\varphi)$ and H are given by Eqs. (1), (2) and (6), respectively.

3. MINIMUM THICKNESS

Eq. (8) does not assume limit thrust line which corresponds to the limit (minimal) thickness of the arch. In order to determine such state, it is neccesary to inspect the flow of the forces (thrust line) through the arch. Thus, the distance δ between the thrust line and the extrados (see Fig. 2c) is defined by:

$$\delta(\varphi) = \rho(\varphi) - \left(R - \frac{t}{2}\right) \tag{9}$$

Analysing the function given by Eq. (9), within the interval between 0 and $\pi/2$, it is determined that its (local) minimal value corresponds to $\varphi=0$. Accordingly, thrust line is closest to the extrados at the summit of the arch; therefore, that position is directive for the minimum thickness value. With respect to Eq. (8), the position of thrust line at the arch's summit, indicated by the point *D* (see Fig. 2a,c), is defined by the following expression:

$$\rho(\varphi \to 0) = \frac{1}{\dfrac{2}{2\,R+t} + \dfrac{12\,R}{12\,R^2 + t^2}} \tag{10}$$

Accordingly, thrust line does not start downwards from the apex of the arch (point *B*), but from the point *D* (being at the certain distance, defined by Eq. (10), from the point *B*).[2] Further, equalizing Eq. (9) with zero (or equalizing Eq. (8) with *R-t/2*), and solve for *t*, results in the following expression:

[2] Milankovitch [7] reached the similar conclusion regarding the masonry buttress with the horizontal force acting at its top.

$$t = \sqrt[3]{2\sqrt{37}R^3 + 11\,R^3} - \frac{3\,R^2}{\sqrt[3]{2\sqrt{37}R^3 + 11\,R^3}} - R \tag{11}$$

For unit value of radius ($R=1$), the value of minimum thickness $t/R_{\min}$ of draped arch (presented by the thickness to radius ratio) is obtained:

$$t/R_{\min} = \sqrt[3]{2\sqrt{37} + 11} - \frac{3}{\sqrt[3]{2\sqrt{37} + 11}} - 1 = 0{,}79829 \approx 4/5 \tag{12}$$

Draped arch of such proportions (having minimum thickness), with the corresponding limit thrust line, is shown in Fig. 3b. In addition, draped arch of the thickness greater and smaller than the minimum is shown in Fig. 3a and Fig. 3c, respectively.

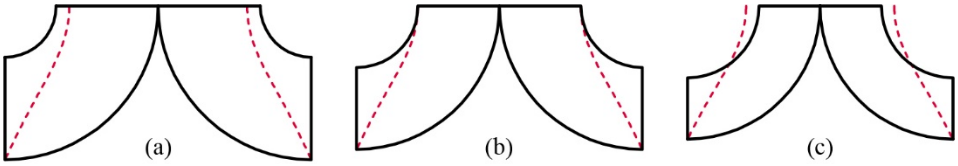

Figure 3. (a) Draped arch of sufficient thickness – stable arch (t/R = 1), (b) minimum thickness and corresponding limit thrust line – limit equilibrium state (t/R = 0,79829), (c) thickness smaller than the minimum – impossible state (t/R = 0,6)

4. CONCLUSION

In this paper, after quilibrium (static) approach, thrust line analysis of draped arch under its own weight has been conducted. Its particular geometric properties have been analytically treated, the coresponding horizontal thrust has been determined, and the closed-form expression of a thrust line has been derived. Moreover, limit equilibrium state is considered, and analysing the flow of the thrust line within the draped arch, the analytical solution for the minimum thickness has been determined. The analysis carried out in this paper can be used as the basis for the appropriate analysis of mechanical behaviour of other more complex inflectional shapes of masonry arches, such as bell-shaped or ogee arches, containing a draped arch as a part. Regarding the further possible developments of the present research, different types of stereotomy, material properties, as well as different loading conditions remain to be considered.

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БЕЛЕШКА О ИСПИТИВАЊУ РАВНОТЕЖЕ МАСИВНОГ ДРАПЕРАСТОГ ЛУКА

Резиме: *Приликом испитивања равнотеже масивних лукова обично се разматрају конвексни облици, као што су кружни, елипсасти или преломљени. Међутим, поред њих постоје и различити конкавни или инфлексиони облици лукова. Стога је циљ овога рада примена теорије потпорне линије на основни лук конкавног облика – драпераст лук, који је образован од двају наспрамно постављених исечака кружног прстена, супротно оријентисаних у односу на полукружни лук. У складу с примењеном нормалном стереотомијом, која подразумева пресеке управне на осу лука, проблем равнотеже под сопственом тежином се аналитички обрађује, те се изводи израз за потпорну линију у затвореном облику. Штавише, на основу разматрања граничног стања равнотеже, одређује се аналитичко решење за минималну теоријску дебљину драперастог лука.*

Кључне речи: *Потпорна линија, масивни лукови, драпераст лук, минимална дебљина, статичко испитивање, испитивање граничног стања равнотеже*