AN APPROXIMATE ANALYTICAL SOLUTION TO LARGE DEFLECTIONS OF A CANTILEVER ROD

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Summary: *This paper deals with the deflection of a cantilever rod bent by a uniform load and a concentrated force at its end point, which involves geometrical nonlinearity and the classical Bernoulli-Euler plane elastica theory. The nonlinear two-point boundary value problem describing the equilibrium configuration of that rod was solved by use of the Laplace transform and the method of successive approximations. The obtained analytical approximation of the solution was compared with both numerical and experimental ones obtained in the laboratory.*

Keywords: *Laplace transform, nonlinear boundary value problems, successive approximation, plane elastica*

1. INTRODUCTION

It is a well known fact that when compared to the theory of finite deformations, the linear theory called strength of materials yields more severe predictions of maximal deflections. For example see [1], Section 16. dealing with a horizontal cantilever rod with a concentrated vertical load acting at its free end, where the solution of corresponding nonlinear two point boundary value problem was given in terms of elliptic integrals. In order to solve more complex problems of plane elastica i.e. problems with different constitutive axioms and different load, engineering communities are still interested in efficient methods for solving both linear and nonlinear differential equations, see [2-8]. Recently, due to an enormous and wide-spread availability of computational power one more efficient method was added to the list, see [9] where the Laplace transform and the method of successive approximations (LT&MSA for short), was used in finding the analytical approximative solutions describing Toda oscillators

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and the optimal orbital transfer. In the latter case the nonlinear two-point boundary value problem (shortly TPBVP) within the standard procedure suggested by the Pontryagin maximum principle was solved. Namely, it is a widely held and somehow misleading belief that the Laplace transform method is particularly suited to solving only linear initial value problems. Regarding TPBVP one may either use the finite Laplace transform, see [10,11] or LT&MSA within the framework of shooting method, see [9], [12]. The novelty of the result that follows will be use of LT&MSA in obtaining the approximate analytical solution of the nonlinear two-point boundary value problem describing the equilibrium configuration of the horizontal cantilever rod bent by a uniform vertical load and a vertical concentrated force acting at its end point. In the following calculations the Mathematica system, release 11.00, and Fujitsu Celsius M470 were used. Finally the results will be compared to numerical ones obtained by ANSYS 5.7.1 software as well as the experimental measurements reported in [13].

2. THE PROBLEM

Consider a linearly elastic rod of length L, straight, prismatic and horizontal in an undeformed state, clamped at the left end, say O, and loaded by a uniformly distributed vertical load along its length, say $w = W / L$, and a vertical concentrated force, say F acting at its free end. Then define the rectangular Cartesian coordinate system $\bar{x}O\bar{y}$ whose axis $\bar{x}$ coincides with the rod axis in the undeformed state and $\bar{y}$ axis perpendicular to $\bar{x}$ axis and oriented downward as the force F, see Fig. 1.

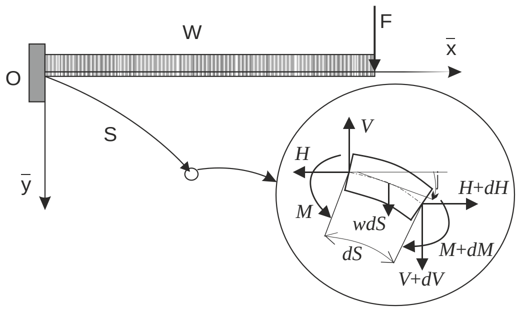

Figure 1 System under consideration

Let EI and S denote bending rigidity of the rod and the arc length of the rod axis measured from O respectively and let and $\varphi = \varphi(S)$ be the angle between the tangent to the rod axis and the $\bar{x}$ axis. Introducing $x = x(S)$ and $y = y(s)$ as the coordinates of an arbitrary point of the rod in the deformed state, $H = H(S)$ and $V = V(S)$ as the components of the resultant force and $M = M(S)$ as the resultant couple, and referring to general theory [14] as well as to Fig. 1, one may state the equilibrium and geometrical equations corresponding to the rod element of length dS,

$$\frac{dH}{dS}=0,\quad \frac{dV}{dS}=-w,\quad \frac{dM}{dS}=-V\cos\varphi,\quad \frac{dx}{dS}=\cos\varphi,\quad \frac{dy}{dS}=\sin\varphi. \tag{1}$$

The constitutive axiom corresponding to the classical Bernoulli-Euler theory reads

$$M=EI\varphi', \tag{2}$$

where $(\cdot)'=d(\cdot)/dS$. The boundary conditions corresponding to this problem read

$$H(L)=0,\ V(L)=F,\ M(L)=0,\ x(0)=0,\ y(0)=0,\ \varphi(0)=0, \tag{3}$$

From the above equations one easily finds the nonlinear TPBVP determining the large deflections of the cantilever rod. It reads

$$EI\varphi''=-[F+w(L-S)]\cos\varphi,\quad \varphi(0)=0,\ \varphi'(L)=0 \tag{4}$$

and further, after introducing the dimensionless quantities

$$t=\frac{S}{L},\quad p=\frac{FL^2}{EI},\quad q=\frac{W}{F}$$

it becomes

$$\ddot{\varphi}=-p[1+q(1-t)]\cos\varphi,\quad \varphi(0)=0,\ \dot{\varphi}(1)=0, \tag{5}$$

where dot over the symbol denotes the derivative with respect to t.

Finally, in order to compare the approximate solution of this problem with numerical one as well as with the measurements of [13] one may choose the following values

$$L=0.4\ \text{m},\quad W=0.3032\ \text{N},\quad EI=0.02591\ \text{Nm}^2,\quad F=0.196\ \text{N}. \tag{6}$$

With this preparation done one may apply LT&MSA to (5). Namely, as in [9], [12] one may introduce the missing initial value, say

$$\dot{\varphi}(1)=\mathrm{b} \tag{7}$$

and then following the lines of [15] apply LT&MSA to find the approximate analytical solution of the corresponding nonlinear initial value problem

$$\ddot{\varphi} = -p[1 + q(1-t)]\cos\varphi\,, \quad \varphi(0) = 0,\ \dot{\varphi}(0) = \mathrm{b}, \tag{8}$$

say $\varphi(t, b)$. Then one may differentiate that solution and apply the Newton method to

$$\dot{\varphi}(1, b) = 0,$$

in order to find the value of b ensuring (5) and (8) to have the same solution.

3. THE SOLUTION

Assuming $\varphi(S) < 1$ and noting that

$$\cos\varphi = 1 - \frac{1}{2}\varphi^2 + \frac{1}{24}\varphi^4 - \frac{1}{720}\varphi^6 + O(\varphi^8),$$

one may use $1 - \varphi^2/2 + \varphi^4/24 - \varphi^6/720$ as a reasonable replacement of cosine function in (5) and (8) so the Cauchy problem to be solved reads

$$\ddot{\varphi} = -p[1 + q(1-t)]\left(1 - \frac{1}{2}\varphi^2 + \frac{1}{24}\varphi^4 - \frac{1}{720}\varphi^6\right),$$
$$\varphi(0) = 0, \qquad \dot{\varphi}(0) = \mathrm{b},$$

Note that the right hand side of the above equation consists of a linear part and a nonlinear part left in curly brackets, i.e.

$$\ddot{\varphi} = pqt - p(q+1) + \{p(q+1-qt)\left(\frac{1}{2}\varphi^2 + \frac{1}{24}\varphi^4 - \frac{1}{720}\varphi^6\right)\},$$
$$\varphi(0) = 0,\ \dot{\varphi}(0) = \mathrm{b}, \tag{9}$$

Then applying the Laplace transform to (9) with $\Phi(S) = \mathcal{L}\{\varphi(t)\}$ one gets

$$\Phi(S) = \frac{b}{s^2} + \frac{p(q+1)}{s^3} + \frac{pq}{s^4} + \frac{1}{s^2}\mathcal{L}\left\{p(q+1-qt)\left(\frac{1}{2}\varphi^2 + \frac{1}{24}\varphi^4 - \frac{1}{720}\varphi^6\right)\right\}.$$
(10)

According to Gustav Doetsch [15, p.106], in order to get the initial approximation of the solution, say $\varphi_0 = \varphi_0(t, b)$ one has to neglect the nonlinear part of (10) and perform the inverse transform, say $\mathcal{L}^{-1}$ so

$$\varphi_0(t, b) = bt - \frac{p(q+1)t^2}{2} + \frac{pqt^3}{6}.$$

Referring to the same reference again the next approximation say $\varphi_1(t, b)$ reads $\varphi_1(t, b) = \varphi_0(t, b)bt +$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{p(q+1-qt)\left(\frac{1}{2}\varphi_0^2(t,b) - \frac{1}{24}\varphi_0^4(t,b) + \frac{1}{720}\varphi_0^6(t,b)\right)\right\}. \tag{11}$$

Note that both the Laplace transform and its inverse in (11) can be easily performed since $\varphi_0(t,b)$ is given in the form of polynomial so $\varphi_0^2(t,b)$, $\varphi_0^4(t,b)$, $\varphi_0^6(t,b)$, will be polynomials too. Also note that $\varphi_1^2(t,b)$, represents polynomial of six degree with respect to *b* and of 19-th degree with respect to t. The continued application of this process lead to

$$\varphi_n(t,b) = \varphi_0(t,b) + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{p(q+1-qt)\left(\frac{1}{2}\varphi_{n-1}^2(t,b) - \frac{1}{24}\varphi_{n-1}^4(t,b) + \frac{1}{720}\varphi_{n-1}^6(t,b)\right)\right\} \quad (12)$$

so in the case of sufficiently regular nonlinear part of the original equation, the solution is found as $\lim_{n\to\infty}\varphi_n(t,b)$ In practice only a few approximations are to be calculated and then one sees whether they are approaching a limit, what will happen only within a certain interval of time, in the present case [0,1].

4. THE RESULTS

For the declared values of the rod and load parameters (6), one may assume that $\varphi_1(t,b)$ will be good enough approximation in this problem. So differentiating $\varphi_1(1,b)=0$ what can be easily done in the Mathematica system and solving $\dot{\varphi}_1(t,b)$ one gets that the solutions of (5) and (8) are the same for

$$b=1.88141$$

For comparison the corresponding value of the numerical solution presented in [13] reads 1.86314. In Fig. 2 one can see agreement of $\varphi_1(t,1.88141)$ with the numerical solution obtained in [13].

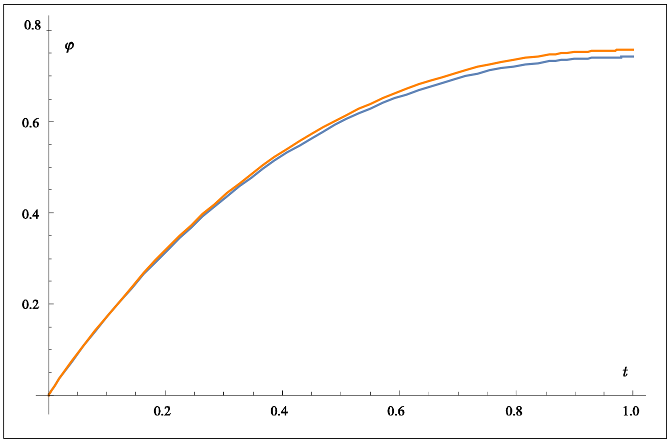

Figure 2 Comparison of approximate (orange) and numerical (blue) solutions of (8)

Finally, referring again to (13) we note that the maximal deflection of cantilever rod $y^N(L)$ obtained from numerical solution reads 0.196. The corresponding value measured in the experiment reported in [13] is $y^e(L) = 0.195$. Note that

$$y^A(L) = 0.4 \int_0^1 \sin \varphi_1(t, 1.88141) dt = 0.213,$$

so the absolute value of the measured value y^e and $y^A(L)$ predicted by the approximate analytical solution (11) for $b = 1.88141$ is less then 0.02 what seems to be good enough. One may speculate that either more terms in the Taylor series of cosine function or the next approximation to be calculated by (12) may increase the accuracy, but all that goes beyond this work.

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ПРИБЛИЖНО РЕШЕЊЕ ЗА КОНАЧНЕ УГИБЕ ТЕШКЕ КОНЗОЛЕ СА СИЛОМ НА КРАЈУ

Резиме:*У раду се проучава угиб конзоле под дејством униформно распоређеног оптерећења и концентрисане силе која делује на њеном слободном крају, а што укључује нелинеарност геометријског типа и класичну Бернули-Ојлерову теорију еластичних штапова. Нелинеарни двотачкасти гранични задатак који описује равнотежну конфигурацију конзоле решаван је применом Лапласових трансформација и метода сукцесивних апроксимација. Добијена аналитичка апроксимација решења граничног проблема упоређивана је са нумеричким решењем као и са експерименталним резутатима добијеним у лабораторији*

Кључне речи: *Лапласова трансформација, нелинерни проблем граничних вредности, сукцесивне апроксимације, раванска теорија еластичних штапова*