THE NEXT STAGE IN NUMERICAL ANALYSIS - AN OVERVIEW OF MESHFREE METHODS

Stepa Paunović[1]
Ivan Nešović[2]
Miloš Milić[3]
Todor Vacev[4]

UDK: 519.673:624.042
DOI:10.14415/konferencijaGFS2017.037

Summary: *Finite element method (FEM) has been successfully used for solving a wide range of engineering problems. However, in some cases the application of FEM is impractical or even impossible. Therefore new methods have been invented that do not need a mesh of elements, but rather rely on approximating the field variable by a set of nodal values - Meshfree (M-Free) or Meshless methods. This paper presents a short overview of the concepts and types of M-free methods, bringing engineers' attention to the new possibilities that they provide.*

Keywords: *Meshfree, M-Free method, FEM, crack modelling, EFG, PIM*

1. INTRODUCTION

Many natural phenomena can be described in terms of algebraic, differential or integral equations. However, exact, analytical solutions to these equations are hard, and often even impossible to obtain, except for the relatively small number of simple and idealized problems. Thus engineers and scientists need to rely on various numerical procedures for finding approximate solutions to the problem of interest. One of the most widely used and most developed numerical procedures for solving differential equations is the Finite Element Method (FEM). FEM is robust, very widely applicable and efficient technique, and it has been used in solving engineering problems with great success over the last couple of decades. Nevertheless, this method has some shortcomings that are becoming more and more apparent [1].

For instance, FEM relies on representing the domain of interest by the *mesh* of elements, but in order to get valid results, elements should have as regular shape as possible, and there are no algorithms that could automatically generate a quality mesh for problems with complex geometries, so the analyst has to spend a lot of time in creating an adequate mesh.

[1]Stepa Paunović, master inž. građ., University of Niš, Faculty of Civil Engineering and architecture, Aleksandra Medvedeva 14, Serbia, tel: ++381 63 1995661, e – mail: stepa.paunovic@gmail.com
[2]Ivan Nešović, master inž. građ., University of Niš, Faculty of Civil Engineering and architecture, Aleksandra Medvedeva 14, Serbia, e – mail: ivan.m.nesovic@gmail.com
[3]Miloš Milić, master inž. građ., University of Niš, Faculty of Civil Engineering and architecture, Aleksandra Medvedeva 14, Serbia
[4]Todor Vacev, dr inž. građ., University of Niš, Faculty of Civil Engineering and architecture, Aleksandra Medvedeva 14, Serbia

Moreover, if an adaptive analysis is required, the mesh has to be refined at each step and there has to be a certain mapping between the meshes, which requires additional computation and greatly increases the (numerical) cost of re-meshing. There are also limitations in analysis of some problems, especially ones that include discontinuities - the crack growth analysis, for instance. It is very difficult to simulate the breaking of the material with the conventional FEM because of its formulation, since it is based on Continuum Mechanics. Another consequence of the FEM formulation procedure is the low accuracy in secondary and derivative variables, such as stresses in Solid Mechanics analysis. For stresses to be continuous and more accurately predicted, either more complex elements and finer meshes should be constructed, or some special techniques (such as the use of super-convergence points or patches) have to be used.

This is why new numerical procedures are being developed. Some of them aim at improving the conventional FEM (Extended FEM - XFEM, General FEM - GFEM, FEM with embedded discontinuities, to name a few), and some have taken new, different approach and have been established as independent procedures. One class of these methods does not require mesh of elements to represent the problem domain, but rather uses only a set of scattered nodes, and thus these are called *Mesh-free* (*MFree*) or *Meshless* methods. There are many different types of MFree methods and the purpose of this paper is to give the reader a short insight into the underlying basic concepts and overview of MFree methods developed so far, in order to bring his or her attention to the new possibilities these methods provide for solving various engineering problems.

2. WHAT ARE MESH-FREE METHODS?

As briefly mentioned above, Mesh-free methods use scattered nodes to represent the problem domain and do not require mesh of predefined elements. Examples of 2D and 3D problem domain representetaion using FEM and MFree method can be seen in *Figure 1*. Since there are lots of similarities between FEM and MFree methods, perhaps the best way to explain the basic ideas of the latter is to contrast them to the well-known concepts of FEM, pointing out and describing the differences.

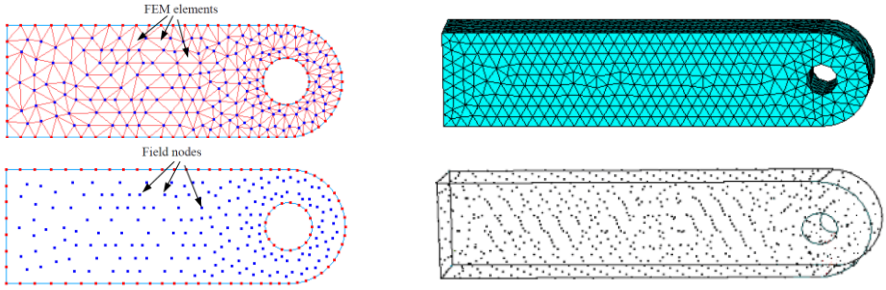

Figure 1. FEM and MFree problem domain representation; 2D problem (left) and 3D problem (right) [1]

The idea behind the FEM

The basic idea in FEM is to devide the problem domain into a number of subdomains and aproximate the sought field variable function over each of the subdomains with the predetermined interpolation function. These subdomains are called *finite elements* and the most commonly used interpolation function is of the form:

$$u(x) \approx u^h(x) = \sum_{i=1}^{n} B_i(x)\alpha_i = \sum_{i=1}^{n} p_i(x)\alpha_i = \mathbf{p}^{\mathrm{T}}\boldsymbol{\alpha} \tag{1}$$

where $u(x)$ is the field variable function that is sought, i.e. the exact solution to the given differential equation, $u^h(x)$ is the aproximate solution to the problem, $B_i(x)$ are the basis functions, α_i are constant coefficients that need to be determained, *n* is the number of terms in series, and also the number of nodes of the finite element at hand. For many elements, polinomials $p_i(x) = x^{i-1}$ are used as the basis functions for interpolation. The last term to the right is matrix form of the above equation, where **p** is the vector of polinomial basis functions, and **α** is the vector of yet unknown coefficients. These coefficients are determained by writing this equation for every node of the element, which yields a system of *n* algebraic equations with *n* unknown constants that, if we adopt the symbol $u_i \equiv u(x_i)$, can be expressed as follows

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} p_1(x_1) & \dots & p_n(x_1) \\ \vdots & \ddots & \vdots \\ p_1(x_n) & \dots & p_n(x_n) \end{bmatrix} \begin{bmatrix} \alpha_i \\ \vdots \\ \alpha_n \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{U} = \mathbf{P}_m\boldsymbol{\alpha} \tag{2}$$

Solving the sistem for **α** and returning these values in *Equation 1* we get

$$u(x) \approx u^h(x) = \mathbf{p}^{\mathrm{T}}\mathbf{P}_m^{-1}\mathbf{U} = \mathbf{N}\,\mathbf{U} = \sum_{i=1}^{n} N_i(x)u_i \tag{3}$$

Here, $N_i(x)$ is the shape function for the node *i* and u_i is the value of the field variable in that node. Since the number of element nodes equals the number of terms in the series used for aproximation, shape functions have the so called δ-function property, i.e. they are unity at their node, and zero in every other node. Consequently, the nodal values of the aproximate solution are in fact the exact values of the field variable, and the aproximate and the exact solutions differ only between the nodes. Such type of aproximation is called *interpolation*[5].

So, for a given problem and the chosen element type, the shape functions $N_i(x)$ are chosen *in advance*, thus conditioning the order of interpolation function, and thereafter the finite elements of the chosen type are arranged into a "mosaic" such as to cover all of the problem

[5] In difference to other types of aproximation where the aproximation function is of lower order and does not pass through every nodal value exactly. The latter type will be termed just *aproximation* in this paper.

domain, and the union of all the interpolation functions of indiidual elements enlarged u_i times at node i gives the aproximate "picture" of the field variable function. It should be mentioned that, if natural or baricentric coordinates and isoparametric formulation for finite elements are used, shape functions are *the same* for each element of the same type.

To ilustrate the aforedescribed procedure, the *Figure 2a* shows how FEM is used to aproximate the solution $u(x)$ over the domain Ω . The coarse mesh is deliberatly chosen for the purposes of illustration. The more accurate aproximation could be achieved either by using more elements (h refinement) or by using elements with higher order of interpolation (p refinement), which would require inserting the so-called interior nodes.

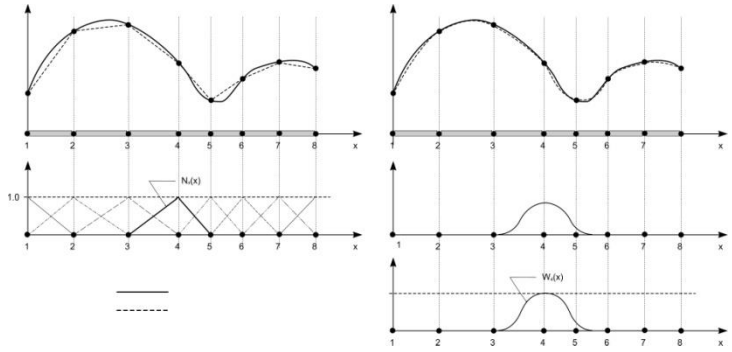

Figure 2. Illustration of FEM interpolation (a), and MFree aproximation (b) functions

Defining the interpolation function $u^h(x)$ over the whole domain requires that first the nodal values u_i be determained. This is done using the constitutive realations to formulate a sistem of equations first for each finite element, and then assambling all these equations into a global system of equations for the whole problem. The most convenient way to do this is the Galerkin weighted residual method, which gives the system of equations of the form

$$\mathbf{KU} = \mathbf{F} \tag{4}$$

where **K** is the global stiffness matrix, **U** is the vector of all nodal displacements and **F** is the vector of nodal loads. In this case, **K** is symmetric and positive definite, and elements of **K** can be determined as

$$K_{IJ}^{e} = \int_{\Omega_e} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\, d\Omega \tag{5}$$

where **D** is the flexibility matrix, and **B** is the matrix of derivatives of the interpolation functions **N**. For different types of problems, **D** will have different elements, and order of **D** and **B** would change, but the form of the *Equation 5* remains the same. It can be concluded that for the most of elements, stiffness matrix has to be determined using

numerical integration (Gauss quadrature for example), which requires background integration cells and a set of integration points.

The idea behind the MFree methods

In these methods the problem domain Ω is represented only by the set of scattered nodes, and the field variable is aproximated *at each point of interest*, using the values in nodes in proximity of that point. The neighbourhood in which nodal values are used for obtaining the aproximation function is called the Local Support Domain (LSD), and the size of LSD is determined by parameters predefined by the analyst. Chosing the appropriate size of LSD is crucial for obtaining accurate and stable results, and it's role somewhat resembles the role of the element size in FEM [2]. LSD can be of any shape, but it is most often set to be circular or rectangular (*Figure 3*).

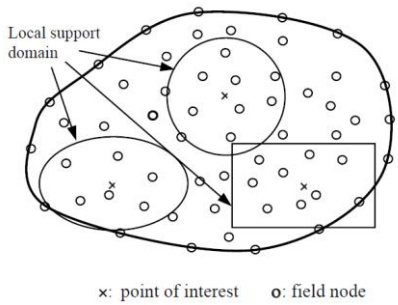

Figure 3 - Various shapes of Local Support Domains [1]

If we use the same methodology of field variable aproximation as in FEM, we get a similar equation which has exactly the same form as *Equation 1*, except now *n* denotes the number of nodes in LSD for the point of interest *x*. This is significant for two reasons. Firstly, this means that aproximation functions are determined *at every point of interest independantly*. Secondly, it also means that the order of the aproximation function depends directly on the number of nodes covered by the LSD. Since there can be a large number of nodes in LSD, this would result in unnecesserily high order of aproximation function, and that is why the following aproximation is used

$$u(x) \approx u^h(x) = \sum_{i=1}^{m} B_i(x)\alpha_i = \sum_{i=1}^{m} p_i(x)\alpha_i = \mathbf{p}^{\mathrm{T}}\boldsymbol{\alpha} \tag{6}$$

where *m* is the pre-chosen, desired order of aproximation function, and $m < n$ always. This, however, leads to a problem in defining the shape functions. Namely, if we follow similar procedure as in FEM and write this equation for each of the *n* nodes in LSD we get the system of equations

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} p_1(x_1) & \dots & p_m(x_1) \\ \vdots & \ddots & \vdots \\ p_1(x_n) & \dots & p_m(x_n) \end{bmatrix} \begin{bmatrix} \alpha_i \\ \vdots \\ \alpha_m \end{bmatrix} \Leftrightarrow [\mathbf{U}]_{n\mathrm{x}1} = [\mathbf{P}_m]_{n\mathrm{x}m}[\boldsymbol{\alpha}]_{m\mathrm{x}1} \tag{7}$$

and now the moment matrix $\mathbf{P}_m$ cannot be inverted to express $\boldsymbol{\alpha}$ in terms of nodal values **U**. In other words, this is not interpolation, but only aproximation of field variable and nodal values cannot be matched exactly. Instead, some procedure is utilised to find "the best aproximation fuction", i.e. the function that diverges minimally from the exact solution. To this end, the Weighted Least Square (WLS) method is often used, and it is based on defining the norm

$$J = \sum_{i=1}^{n} W_i (u^h(x_i) - u_i)^2 \tag{8}$$

and then minimizing it by finding the derivatives $\partial J / \partial \alpha_i$. This yields the system of m equations that can be solved for m unknown coefficieants α_i :

$$\mathbf{P}_m^{\mathrm{T}} \mathbf{W}\, \mathbf{P}_m\, \boldsymbol{\alpha} = \mathbf{P}_m \mathbf{W}\, \mathbf{U} \qquad \Leftrightarrow \qquad \mathbf{A}\, \boldsymbol{\alpha} = \mathbf{B}\, \mathbf{U} \tag{9}$$

Here **W** is a diagonal matrix of nodal weights W_i for nodes in LSD, calculated for every point of interest independently, using some bell-shaped weight function. The purpose of these weights is to include the influence of proximity of a given node to the point of interest - the closer the node, the higher its weight. There are many types of weight functions that can be used, but they are usually such as to equal unity at the point of interest (that is, at the center of LSD), and then gradually decline towards the border of LSD, finally reaching zero outside the LSD. This enables the later derived global stiffness matrix to be spars and thus the system of equations effectivly solved. Now the shape functions can be expressed as

$$u(x) \approx u^h(x) = \mathbf{p}^{\mathrm{T}} \mathbf{A}^{-1} \mathbf{B}\, \mathbf{U} = \boldsymbol{\Phi}\, \mathbf{U} = \sum_{i=1}^{n} \Phi_i(x) u_i \tag{10}$$

Since this is not interpolation, these shape functions do not posess the δ-function property and this makes it harder to impose essential boundary conditions so some special techniques have to be used. In *Figure 3* the aproximation and shape functions of MFree method or some 1D problem are shown. Again, as in FEM, there are only a few nodes for for the purpose of illustration.

If the Solid Mechanics problem is investigated in its weak form, the equations obtaind through Galerkins WRM for example are of exactly the same form as in FEM, namly:

$$\mathbf{KU} = \mathbf{F} \qquad \text{with} \quad K_{IJ} = \int_{\Omega} \mathbf{B}^{\mathrm{T}} \mathbf{D} \mathbf{B}\, d\Omega \tag{11}$$

but now the integration does not happen iside the subdomain of each element Ω_e, but rather at integration points throughout the entire problem domain Ω. Nevertheless, the background cells and a set of integration points are still required for numerical integration (thus this method is not "truly meshless", but there are truly meshless ones and they will be discussed shortly). However, the big difference compared to FEM is that the

aproximation functions need to be determined separately for each point of interest, i.e. interation point in this instance, in contrast to FEM where the interpolation functions are the same and predefined for each element. Therefore, an algorithm for numerical implementation of the illustrated MFree method for solving a Solid Mechanics problem can be summerized as shown in [1].

3. CLASSIFICATION AND TYPES OF MESH-FREE METHODS

The MFree method described in the previous section is just one of many MFree methods developed so far. There are several basis for classification of these methods.

For instance, in the above example, the Galerkin weak-form formulation was used, and the global stiffness matrix elements was calculated by integration over the whole problem domain. This is the so-called Global Weak-Form method. If the domain is devided to a number of subdomains and then the Galerkin transformtion is used, then it's the Local Weak-Fom method. One could also solve the problem in its strong-form, and thus avoid numerically expensive integration. MFree strong-form method is the collocation method and it is very effective but requires some speciall treatments to impose boundary conditions.
In the given example the field variable function was aproximated by a series. This is the series-type representation of the function. There are also the integral representation methods (Smooth Partical Hydrodynamycs, Reproducing Kernel Partical method) and differential representation methods (Collocation method).

Also, the aproximation functions in the example above needed to be determined at each integration point. Methods of this type are called Point-Interpolation methods. There is also a Moving Least Squares aproximation (the most fameous of these methods is the Element Free Galerkin (EFG) method), as well as the methods with some "extermnal" enrichments such as the *hp*-cloud method and Partition of Unity methods.

In addition, the basis functions in the example are chosen to be polinomials of the form $p_i(x) = x^{i-1}$. These are called Polinomial Basis Functions (PBF). Alternatively, one could use the so-called Radial Basis Functions (RBF) for aproximation [1, 2]. So, the method used in this example could be categorized as the Polinomial Point Interpolation method for Global Galerkin weak-form problem formulation. If the RBF were used instead of PBF, that method would be the Radial Point Interpolation method - RPIM.

4. PERFORMANCE COMPARISON OF FEM AND MFree METHODS

In general, MFree methods have better accuracy and rate of convergence then FEM, and ususally require less computation time (depending on the method used), even for some classical problems, let alone the complex ones. Figure 4 shows the accuracy and rate of convergence diagrams for FEM with bi-linear elements and two MFree methods - RPIM and EFG, for one of the standard benchmark problems - a cantilever beam.

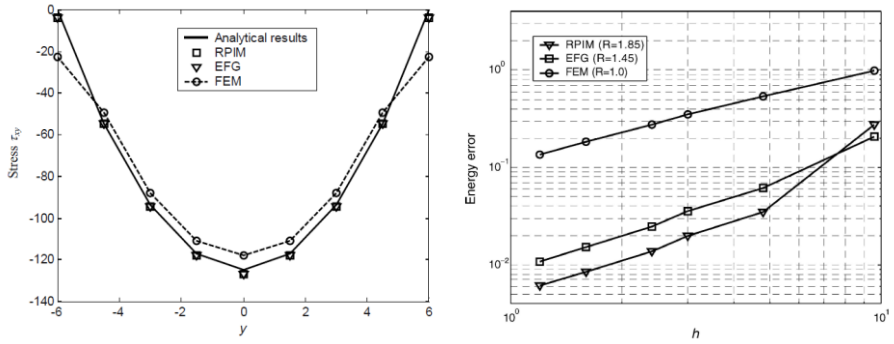

Figure 4. Accuracy (left) and rate of convergence (right) diagrams for FEM, EFG and RPIM, compared to the analitical solution for shearing cross-sectional stresses of a cantilever beam. R is the rate of convergence [1]

It's evident that MFree methods exhibit better accuracy and rate of convergence than FEM. Moreover, they are somewhat faster than the conventional FEM, which can be seen from the Figure 5, where the CPU time needed for solution is plotted.

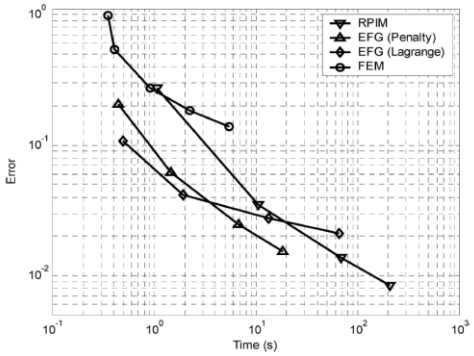

Figure 5. The CPU time for convetional FEM, RPIM and EFG with two different ways of imposing the boundary conditions (penalty mode and Lagrange multipliers) [1]

5. POSSIBILITIES AND FUTURE DEVELOPEMENT

Compared to conventional FEM, MFree methods give better results in terms of accuracy and convergence when applied to certain classes of problems. However, depending on the formulation used, system matrices produced by these methods can lack symmetry and be highly populated (not sparse) which makes the solving harder and more costly. Therefore, one of the tasks in the future should be improving the existing formulations and defining the new ones such as to be more numerically effective.

In every MFree method, one of the most important parts is the choice of basis functions and weight functions, since they directly influence the aproximation capabilities and

numerical stability of the solution. Figure 6 shows how different basis functions affect the final aproximation function. So far only the RBF and PBF have been used, and there is a lot of space for developing new basis functions that would perform better.

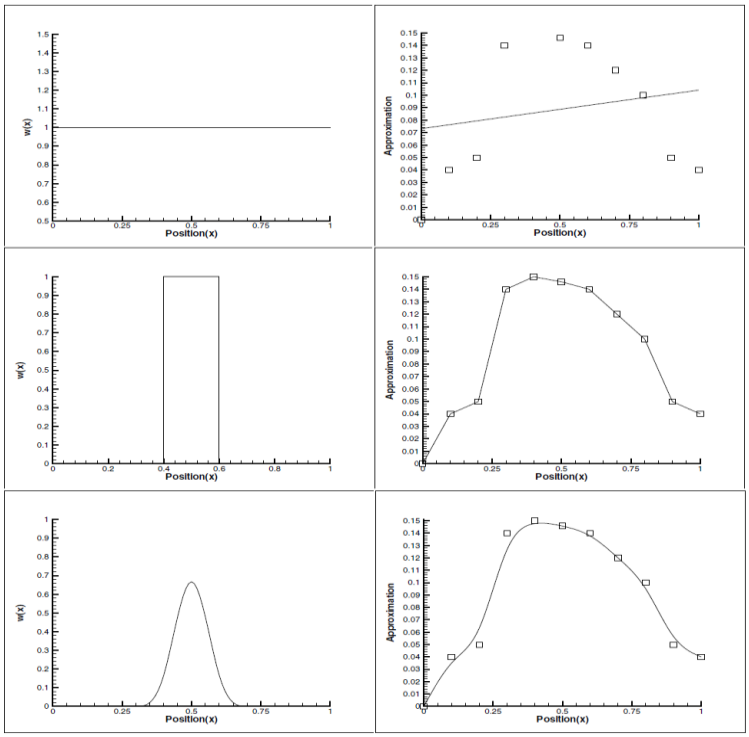

Figure 6. The effect of different weight functions (on the left) on the aproximate solution function (on the right) using the MLS formulation in EFG method [3]

Since both FEM and MFree methods have their advantages, as well as shortcomings, some authors combine them to get hybrid models getting the most out of both methods. As an example of this, in Figure 7 is shown how FEM and EFG can be combined in crack propagition analysis [4,5].

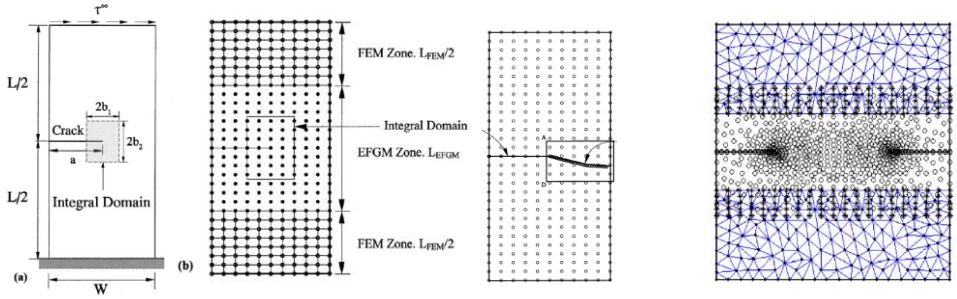

Figure 7. Combination of FEM and EFG for crack propagition analysis [5,6]

6. CONCLUDING REMARKS

In the light of everything said, it can be concluded that Mesh-free methods present a promising and valuable tool for solving wide range of engineering problems. However, they should not be regarded as the supstitute for FEM, but rather its usefull compliment, and engineers versed in both methods could combine them to form even more efficient numerical procedures, or use the more suitable one for the problem at hand to obtain more reliable and accurate results. FEM is very thoraughly developed, and MFree methods are still in their developement stage, but they already show that they are a challenging and rich field to explore, certainly worth the attantion.

REFERENCES

[1] Liu, G.R., Gu, Y.T.: *An Introduction to Meshfree Methods and Their Programming*, Springer, Dordrecht, **2005** *Materijali i konstrukcije*, **2003.**, vol. 46, № 1-2, p.p. 5-15.
[2] Nguyen, V.P., Rabczuk, T., Bordas, S., Duflot, M.: Meshless methods: a review and computer implementation aspects. *Mathematics and computers in simulation*, **2008**, vol. 79, pp. 763-813
[3] Dolbow, J., Belytschko, T.: An Introduction to Programming the Meshless Element Free Galerkin Method. *Archives of Computational Methods in Engineering*, **1998**, Vol. 5, 3, pp. 207-241
[4] Rao, B.N., Rahman, S.: A coupled meshless-finite element method for fracture analysis of cracks. *International jurnal of Pressuure Vessels and Piping*, **2001**, Vol. 78, pp. 647-657
[5] Gu, Y.T., Zhang, L.C.: Coupling of the meshfree and finite element methods for determination of the crack tip fields. *Engineering Fracture Mechanics*, **2008**, Vol 75, pp. 986-1004

СЛЕДЕЋИ СТУПАЊ У НУМЕРИЧКОМ МОДЕЛИРАЊУ-ПРЕГЛЕД БЕЗМРЕЖНИХ МЕТОДА

Резиме: *Метод коначних елемената (МКЕ) се успешно користи при решавању широког спектра инжењерских проблема. Међутим, у неким случајевима је непрактично или чак немогуће применити МКЕ. Стога су развијене нове методе које не користе мрежу елемената већ поље разматране променљиве апроксимирају само преко њене вредности у чворовима - безмрежне методе, тј. методе без мреже. Овај рад представља кратак преглед концепата и типова безмрежних метода, у циљу скретања пажње инжењера на нове могућности које ове методе пружају.*

Кључне речи: *Безмрежне методе, МКЕ, моделирање прслина*