ON THE POST-CRITICAL BEHAVIOUR OF AN OPTIMALLY SHAPED ELASTIC COLUMN

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Summary: *By using Pontryagin's maximum principle we determine the optimal shape of an elastic column. The column is loaded by a compressive force and positioned on elastic foundation of Winkler type. We assume that column is clamped on one end and elastically supported on the other. The optimality conditions for the case of bimodal optimization are derived. For moderate value of foundation rigidity and support constant we obtained different post-buckling modes.*

Keywords: *stability, elastic foundation, optimal shape, post-critical behavior*

1. INTRODUCTION

The problem of determining the shape of a column, of given volume that is the strongest against buckling, is an important engineering problem. It was formulated by Lagrange [1] and is now known as the *Lagrange problem*. For the historical account of the Lagrange problem see, for example, [2] and [3]. For a column on elastic, Winkler type foundation the problem of determining the optimal shape was treated in [4] and [5].
A *bimodal* optimization procedure was formulated in [6] and [7] for the column without elastic foundation, and with the boundary conditions that we use (both ends clamped). A number of important problems of structural stability are characterized by multiple buckling modes associated with the same critical load. For such problems the initial post-buckling behaviour is considerably more complicated with problems with simple (unimodal) bifurcation points. The stability boundary and post-buckling behavior of an elastic rod with spring supports at clamped ends is determined by Glavardanov and Maretic in [8].Our goal in this work is to determine post-buckling behaviour of an optimally shaped column. Thus, we shall first determine the optimal shape of a column that rests on a linearly elastic (Winkler) foundation with clamped-elastic supported ends and then its post-buckling (deformed) shape.

2. MATHEMATICAL FORMULATION

Consider a column of length L shown in Figure 1. The column is clamped at one end and elastically supported on the other end, with end C having the possibility of sliding along

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the x axis. At the end C the column is loaded by a compressive force F. The column is positioned on a Winkler type of foundation. Equilibrium equations for the column are, the geometrical and constitutive equationare [9]

$$\frac{dH}{dS}=0,\ \ \frac{dV}{dS}=-q_y,\ \ \frac{dM}{dS}=-V\cos\theta+H\sin\theta,$$
$$\frac{d\bar{x}}{dS}=\cos\theta,\ \ \frac{d\bar{y}}{dS}=\sin\theta,\ \ M=EI\frac{d\theta}{dS}. \tag{1}$$

where $q_y= -\mu\, y$ and $\mu > 0$ is a constant stiffness of the foundation, H and V are components of the contact force (i.e. the resultant force in an arbitrary cross-section) along x and y axes, respectively, M is the bending moment, θ is the angle between the tangent to the column axis and x axis of a rectangular Cartesian coordinate system x-B-y and S is the arc-length of the column axis measured from the origin of the coordinate system B.

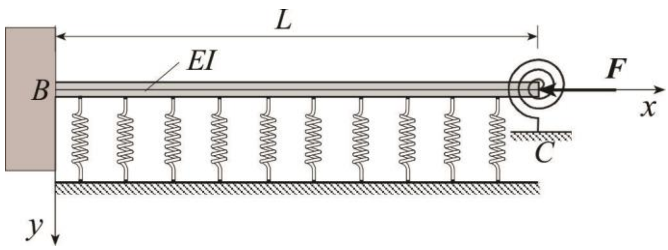

Figure 5. Coordinate system and load configuration

In Eq.(1) we used $\bar{x}$ and $\bar{y}$ to denote coordinates of an arbitrary point on the rod axis in the coordinate system x-B-y, E is modulus of elasticity and I is the moment of inertia of the cross-sectional area. Equations (2), (3) correspond to the classical Bernoulli-Euler rod theory. The boundary conditions for the column shown in Figure 1. are

$$\bar{y}(0)=\bar{y}(L)=0,\ \ \theta(0)=0,$$
$$M(L)=-k_a\theta(L),\ \ H(L)=-F, \tag{2}$$

where k_a is spring constant of the support. Solving Eqs.$(1)_{1,2}$ and by using Eq.$(2)_3$ we obtain $H=-F$.
The volume of the column is

$$W=\int_0^L A(S)\,dS, \tag{3}$$

where A is the cross sectional area. We assume that $I=\alpha A^2$, where α is a constant. There are other possibilities to assume this relations. For a circular cross-section $\alpha=1/4\pi$. By introducing the dimensionless quantities

$$t=\frac{S}{L},\quad a=\frac{A}{L^2},\quad \zeta=\frac{\bar{x}}{L},\quad \eta=\frac{\bar{y}}{L},\quad w=\frac{W}{L^3},$$
$$\lambda_1=\frac{\mu}{\alpha E},\quad \lambda_2=\frac{F}{\alpha E L^2},\quad m=\frac{M}{\alpha E L^3},\quad k=\frac{k_a}{\alpha E L^3} \tag{4}$$

we obtain from Eqs.(1), (2)

$$\dot{v}=\lambda_1\eta,\quad \dot{m}=-v\cos\theta-\lambda_2\sin\theta,\quad \dot{\zeta}=\cos\theta,\quad \dot{\eta}=\sin\theta,\quad \dot{\theta}=\frac{m}{a^2}, \tag{5}$$

subject to

$$\zeta(0)=0,\quad \eta(0)=0,\quad \eta(1)=0,\quad \theta(0)=0,\quad m(1)=-k\theta(1), \tag{6}$$

where $\dot{(\cdot)}=\frac{d}{dt}(\cdot)$. The dimensionless volume becomes

$$w=\int_0^1 a(\xi)\,d\xi. \tag{7}$$

The system (5), (6) has a trivial solution $\theta_0=\eta_0=v_0=0,\ \zeta_0=t$, for all values of load parameter λ_2 and stiffness parameter λ_1. To determine $(\lambda_1,\lambda_2)\in \mathbb{R}^2$ for which there is a nontrivial solution to (9), (10) we write $v=v_0+\Delta v,...,\theta=\theta_0+\Delta\theta$ where $\Delta v,\dots,\Delta\theta$ are perturbations. By substituting this into Eqs. (5), (6) and by neglecting the higher order terms in perturbations, we obtain (after omitting Δ in front of Δv etc.)

$$\dot{v}=\lambda_1\eta,\quad \dot{m}=-v-\lambda_2\theta,\quad \dot{\zeta}=0,\quad \dot{\eta}=\theta,\quad \dot{\theta}=\frac{m}{a^2}, \tag{8}$$

subject to

$$\eta(0)=0,\quad \eta(1)=0,\quad \theta(0)=0,\quad m(1)=-k\theta(1). \tag{9}$$

A necessary condition that system (5), (6) has a nontrivial solution (i.e., loss of stability of the column by buckling) is that (8), (9) has a nontrivial solution. In [10] we showed that the multiplicity of an eigenvalue for the system (8), (9) can be at most two.

We assume that the cross sectional area $a(t)$ belongs to the set $\mathbf{U}$ called the set of *admissible* cross sectional area functions. In what follows we assume that $\mathbf{U}$ is the set of twice continuously differentiable functions on the interval $[0,1]$, i.e., $\mathbf{U}=C^2(0,1)$ that are nonnegative $a(t)\geq 0$.

Suppose now that $(\lambda_1^*,\lambda_2^*)\in \mathbb{R}^2$ is given (for chosen k). We define *the optimal compressed column on an elastic foundation with clamped-elastically supported ends* as

the column so shaped that any other column of same length (in our case equal to one) and smaller volume will buckle under load and foundation characterized by $(\lambda_1^*,\lambda_2^*)$. Thus, the problem of determining the shape of the optimal column may be, mathematically, stated as:
Given $(\lambda_1,\lambda_2)=(\lambda_1^*,\lambda_2^*)$, find $a^*(t)\in\mathbf{U}$ such that the integral (7) is in minimum for $a^*(t)\in\mathbf{U}$ among all those $a(t)\in\mathbf{U}$ such that when $a^*(t)$ is used in Eqs. (8), (9) the values (λ_1,λ_2) determined from Eqs. (8), (9) are equal to $(\lambda_1^*,\lambda_2^*)$.

3. METHOD OF OPTIMIZATION

In order to apply the Pontryagin's maximum principle we introduce new dependent variables as $x_1=\eta,\quad x_2=\theta,\quad x_3=v,\quad x_4=m.$ Then, the system (8), (9) becomes

$$\dot{x}_1=x_2,\quad \dot{x}_2=\frac{x_4}{a^2},\quad \dot{x}_3=\lambda_1 x_1,\quad \dot{x}_4=-x_3-\lambda_2 x_2, \tag{10}$$

and

$$x_1(0)=0,\quad x_1(1)=0,\quad x_2(0)=0,\quad x_4(1)=-kx_2(1). \tag{11}$$

In terms of the optimal control, the Problem now becomes: Given (λ_1,λ_2) find the control $a^*(t)\in\mathbf{U}$ such that $\min_{a\in\mathbf{U}} I=\min_{a\in\mathbf{U}}\int_0^1 a(t)\,dt=\int_0^1 a^*(t)\,dt$ if the system is subjected to (9), (10). Suppose now that for given (λ_1,λ_2) and for the optimal $a(t)=a^*(t)$ the linear boundary value problem (10), (11) has *two* linearly independent solutions, $(\bar{x}_1,...\bar{x}_4)$ and $\hat{x}_1,...\hat{x}_4$ corresponding to two buckling modes. Since both solutions correspond to the same (λ_1,λ_2) and $a(t)=a^*(t)$ we have (see [10])

$$\begin{aligned}&\dot{\bar{x}}_1=\bar{x}_2,\quad \dot{\bar{x}}_2=\frac{\bar{x}_4}{a^2},\quad \dot{\bar{x}}_3=\lambda_1\bar{x}_1,\quad \dot{\bar{x}}_4=-\bar{x}_3-\lambda_2\bar{x}_2,\\ &\dot{\hat{x}}_1=\hat{x}_2,\quad \dot{\hat{x}}_2=\frac{\hat{x}_4}{a^2},\quad \dot{\hat{x}}_3=\lambda_1\hat{x}_1,\quad \dot{\hat{x}}_4=-\hat{x}_3-\lambda_2\hat{x}_2,\end{aligned} \tag{12}$$

satisfying

$$\begin{aligned}&\bar{x}_1(0)=\bar{x}_1(1)=\bar{x}_2(0)=0,\quad \bar{x}_4(1)=-k\bar{x}_2(1),\\ &\hat{x}_1(0)=\hat{x}_1(1)=\hat{x}_2(0)=0,\quad \hat{x}_4(1)=-k\hat{x}_2(0),\end{aligned} \tag{13}$$

To determine $a^*(t)$ we use the standard procedure of Optimal control theory (see [11]). Thus, we form the Pontryagin's function H, taking into account that differential constraints are given by Eqs.(12). Therefore

$$\begin{aligned} \mathrm{H} &= a + \bar{p}_1 \bar{x}_2 + \bar{p} \frac{\bar{x}_4}{a^2} + \bar{p}_3 \lambda_1 \bar{x}_1 + \bar{p}_4 \left(-\bar{x}_3 - \lambda_2 \bar{x}_2 \right) \\ &+ \hat{p}_1 \hat{x}_2 + \hat{p} \frac{\hat{x}_4}{a^2} + \hat{p}_3 \lambda_1 \hat{x}_1 + \hat{p}_4 \left(-\hat{x}_3 - \lambda_2 \hat{x}_2 \right) \end{aligned} \tag{14}$$

where the costate variables $\bar{p}_i, \hat{p}_i, i = 1,...,4$ satisfy

$$\begin{aligned} &\dot{\bar{p}}_1 = -\frac{\partial \mathrm{H}}{\partial \bar{x}_1} = -\bar{p}_3 \lambda_1, && \dot{\bar{p}}_2 = -\frac{\partial \mathrm{H}}{\partial \bar{x}_2} = -\bar{p}_1 + \lambda_2 \bar{p}_4, \\ &\dot{\bar{p}}_3 = -\frac{\partial \mathrm{H}}{\partial \bar{x}_3} = \bar{p}_4, && \dot{\bar{p}}_4 = -\frac{\partial \mathrm{H}}{\partial \bar{x}_4} = -\frac{\bar{p}_2}{a^2}, \\ &\dot{\hat{p}}_1 = -\frac{\partial \mathrm{H}}{\partial \hat{x}_1} = -\hat{p}_3 \lambda_1, && \dot{\hat{p}}_2 = -\frac{\partial \mathrm{H}}{\partial \hat{x}_2} = -\hat{p}_1 + \lambda_2 \hat{p}_4, \\ &\dot{\hat{p}}_3 = -\frac{\partial \mathrm{H}}{\partial \hat{x}_3} = \hat{p}_4, && \dot{\hat{p}}_4 = -\frac{\partial \mathrm{H}}{\partial \hat{x}_4} = -\frac{\hat{p}_2}{a^2}, \end{aligned} \tag{15}$$

subject to

$$\begin{aligned} &\bar{p}_3(0) = 0, \quad \bar{p}_3(1) = 0, \quad \bar{p}_4(0) = 0, \quad \bar{p}_2(1) = k\bar{p}_4(1), \\ &\hat{p}_3(0) = 0, \quad \hat{p}_3(1) = 0, \quad \hat{p}_4(0) = 0, \quad \hat{p}_2(1) = k\hat{p}_4(1), \end{aligned} \tag{16}$$

The optimality condition $\min\limits_{a \in \mathbf{U}} \mathrm{H}$ leads to

$$\frac{\partial \mathrm{H}}{\partial a} = 1 - 2\bar{p}_2 \frac{\bar{x}_4}{a^3} - 2\hat{p}_2 \frac{\hat{x}_4}{a^3} = 0. \tag{17}$$

By solving (17) for *a* we obtain

$$a = a^* = \left[2\left(\bar{p}_2 \bar{x}_4 + \hat{p}_2 \hat{x}_4 \right) \right]^{1/3}. \tag{18}$$

In order to reduce the dimension of the system, we proposed in [2] the identification of state and co-state variables as

$$\begin{aligned} &\bar{p}_1 = \beta_{11}\bar{x}_3 + \beta_{12}\hat{x}_3; && \bar{p}_2 = \beta_{11}\bar{x}_4 + \beta_{12}\hat{x}_4; \\ &\bar{p}_3 = -\beta_{11}\bar{x}_1 - \beta_{12}\hat{x}_1; && \bar{p}_4 = -\beta_{11}\bar{x}_2 - \beta_{12}\hat{x}_2; \\ &\hat{p}_1 = \beta_{21}\bar{x}_3 + \beta_{22}\hat{x}_3; && \hat{p}_2 = \beta_{21}\bar{x}_4 + \beta_{22}\hat{x}_4; \\ &\hat{p}_3 = -\beta_{21}\bar{x}_1 - \beta_{22}\hat{x}_1; && \hat{p}_4 = -\beta_{21}\bar{x}_2 - \beta_{22}\hat{x}_2, \end{aligned} \tag{19}$$

where β_{ij}, $i, j = 1,2$ are constants. Note that with (18) cross sectional area becomes

$$a(t) = a^*(t) = \left[2\left(\gamma_{11}(\bar{x}_4)^2 + 2\gamma_{12}\bar{x}_4\hat{x}_4 + \gamma_{22}(\hat{x}_4)^2 \right) \right]^{1/3}, \tag{20}$$

where $\gamma_{11} = \beta_{11,}$ $\gamma_{12} = (\beta_{12} + \beta_{21})/2,$ $\gamma_{22} = \beta_{22}$.
Also from Eq.(20) it follows that $a(t) \geq 0$. Therefore, we conclude that the optimal shape of the column is determined from (20) when (10), (11) is solved. Thus, the relevant system of equations is

$$\begin{aligned} &\dot{\bar{x}}_1 = \bar{x}_2, \quad \dot{\bar{x}}_2 = \frac{\bar{x}_4}{\left[2\left(\gamma_{11}(\bar{x}_4)^2 + 2\gamma_{12}\bar{x}_4\hat{x}_4 + \gamma_{22}(\hat{x}_4)^2 \right) \right]^{2/3}}, \\ &\dot{\bar{x}}_3 = \lambda_1\bar{x}_1, \quad \dot{\bar{x}}_4 = -\bar{x}_3 - \lambda_2\bar{x}_4, \\ &\dot{\hat{x}}_1 = \hat{x}_2, \quad \dot{\hat{x}}_2 = \frac{\hat{x}_4}{\left[2\left(\gamma_{11}(\bar{x}_4)^2 + 2\gamma_{12}\bar{x}_4\hat{x}_4 + \gamma_{22}(\hat{x}_4)^2 \right) \right]^{2/3}}, \\ &\dot{\hat{x}}_3 = \lambda_1\hat{x}_1, \quad \dot{\hat{x}}_4 = -\hat{x}_3 - \lambda_2\hat{x}_4. \end{aligned} \tag{21}$$

subject to (13).

4. POST-BUCKLING BEHAVIOUR

We now study the initial post-critical behavior of an optimally shaped column. Our solution procedure is as follows:

1) We solve Eqs. (12),(13) for (λ_1, λ_2^*) and chosen value of parameter k to determine $a(t)$. The values of (λ_1, λ_2^*) where so determineted that $w = 1$.
2) With so obtained $a(t)$ we solve Eqs. (5), (6) for $(\lambda_1, \lambda_2, k)$.

4.1.Numerical results

First, we consider column on elastic foundation for $\lambda_1 = 300, \lambda_2^* = 51.8115,$ and $\gamma_{11} = 1,$ $\gamma_{22} = 3.6,$ $\gamma_{12} = 1$. Parameter of spring support constant is $k = 0.1$. Buckling modes are shown in Figure 2.

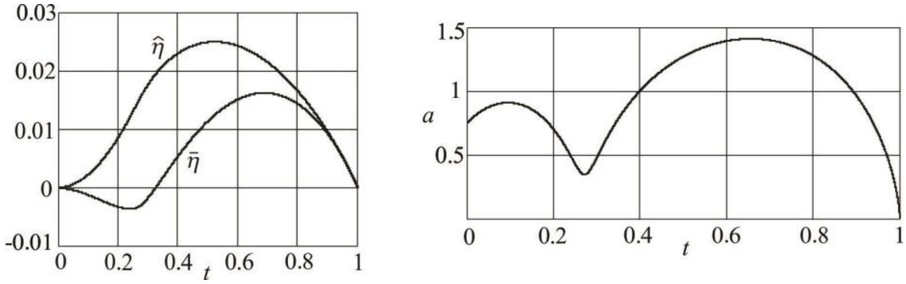

Figure 2. Buckling modes and cross-sectional area of the column

$$\lambda_1 = 300, \lambda_2^* = 51.8115, k = 0.1$$

Cross-sectional area is shown on the right side of Figure 2. Maximum value of cross-sectional area is $a_{\max} = 1.41697719$ and $a(1) = 0.1181631332$. Post–buckling configurations for the case $\lambda_1 = 300,\ k = 0.1$ are shown in Figure 3.

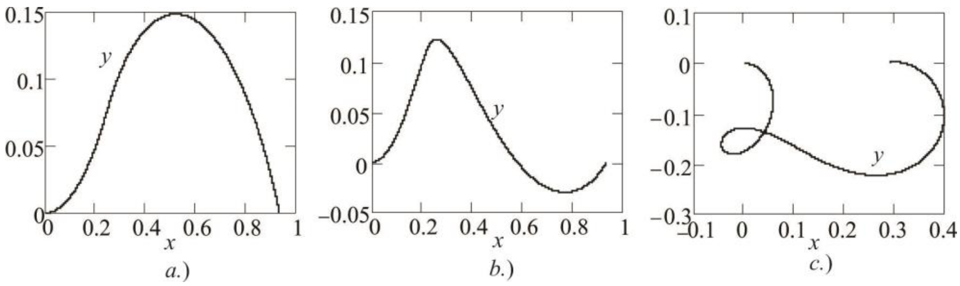

Figure 3. Post-buckling modes for the case $\lambda_1 = 300,\ k = 0.1$

$$a.)\ \lambda_2 = 50,\quad b.)\ \lambda_2 = 52,\quad c.)\ \lambda_2 = 52$$

We obtained post-critial buckling mode for $\lambda_2 = 50$ (showen on Figure 3.a.)) coresponding to first buckling mode.

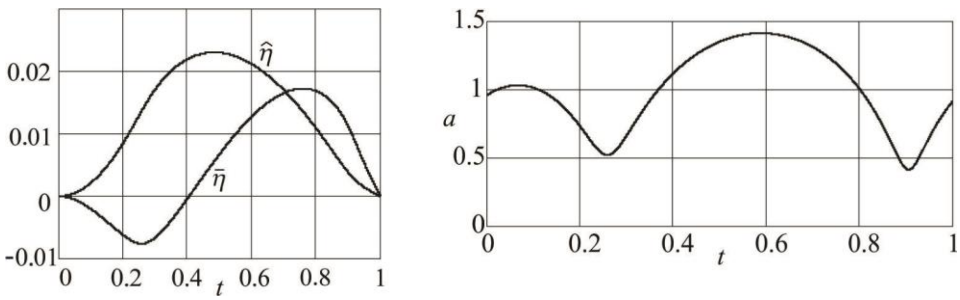

Figure 4. Buckling modes and cross-sectional area of the column

$$\lambda_1 = 300, \lambda_2^* = 62.3075, k{=}5$$

Super-crical buckling mode obtained for $\lambda_2 = 52$ are showen on Figure 3.b.) (coresponding to second mode). Buckling mode on Figure 3.c.) do not bifurcate from the trivial state.

In next example we treated column on elastic foundation with parameter of foundation $\lambda_1 = 300$, parameter of axial force $\lambda_2^* = 62.3075$ and $\gamma_{11} = 1,\ \gamma_{22} = 3.6,\ \ \gamma_{12} = 1$. Parameter of spring support constant is $k = 5$. In this case we have bimodal solution. Buckling modes and cross-sectional area are shown in Figure 4. Maximum value of cross sectional area is $a_{\max} = 1.41435651$. Post-buckling shapes for the case $\lambda_1 = 300,\ k = 5$ are shown in Figure 5.

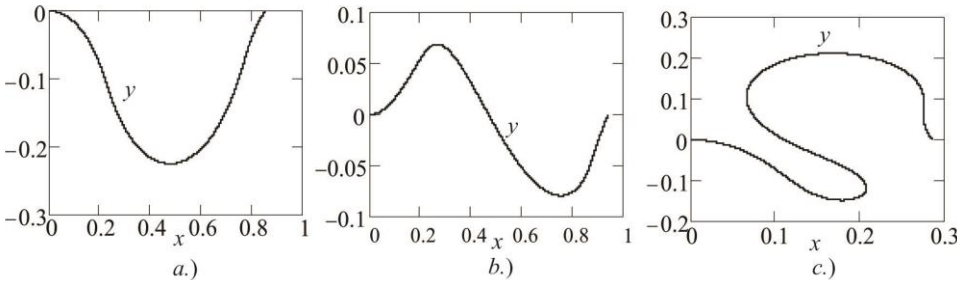

Figure 5. Post-buckling modes for the case $\lambda_1 = 300,\ k = 5$

$a.)\ \lambda_2 = 60,\ \ b.)\ \lambda_2 = 64,\ c.)\ \lambda_2 = 64$

We obtained buckling mode for $\lambda_2 = 60$ (showen on Figure 5.a.)) coresponding to first buckling mode. Super-critical buckling mode obtained for $\lambda_2 = 64$ are showen on Figure 5.b.) . Buckling mode on Figure 5.c.) do not bifurcate from the trivial state.

Last example treat column with no elastic foundation ($\lambda_1 = 0$) for $\lambda_2^* = 38.9622,$ and $\gamma_{11} = 1,\ \gamma_{22} = 3.6,\ \ \gamma_{12} = 1$. Parameter of spring support constant is $k = 5$. Buckling modes and optimal cross-sectional area are shown in Figure 6. Maximum value of cross sectional area is $a_{\max} = 1.40097664$.

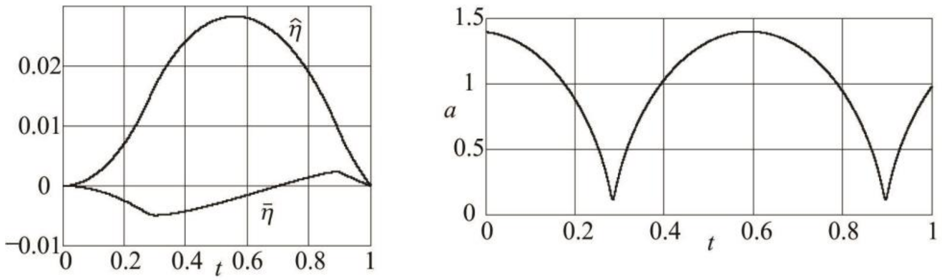

Figure 6. Buckling modes and cross-sectional area of the column

$\lambda_1 = 0, \lambda_2^* = 38.9622, k{=}5$

Post-buckling shapes for the case $\lambda_1 = 0,\ \lambda_2 = 40,\ k = 5$ are shown in Figure 7. We obtained post-buckling shape corresponding to first, second and combined modes.

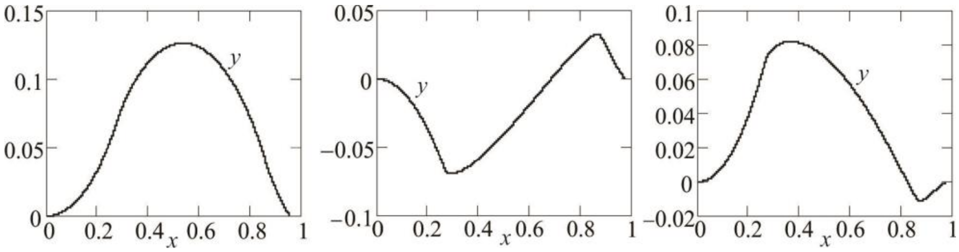

Figure 7. Post-buckling modes for the case $\lambda_1 = 0,\ \lambda_2 = 40,\ k = 5$

5. CONCLUTIONS

We studied the post-critical behaviour of optimally shaped columns on elastic foundation of Winkler type. Our main results may be stated as:

1.) In all cases we have bimodal optimization.
2.) For constant value of foundation rigidity and various value of spring support constant we obtained three different post-buckling modes corresponding to first, second and buckling mode that do not bifurcate from the trivial state. These results are in agreement with the results presented in [13].
3.) For the column with no elastic foundation and $k = 5$ we obtined only super-critical buckling modes coressponding to first, second and combined modes.

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ПОСЛЕКРИТИЧНО ПОНАШАЊЕ ОПТИМАЛНО ОБЛИКОВАНОГ ЕЛАСТИЧНОГ ШТАПА

Резиме: *Коришћењем Понтријагиновог принципа максимума одређен је оптимални облик еластичног штапа. Штап је оптерећен аксијалном силом притиска и налази са на еластичној подлози Винклеровог типа. Посматрамо штап уклештен на једном и еластично ослоњен на другом крају. Изведени су услови оптималност за случај бимодалне оптимизације. За различите вредности крутости подлоге и ослоначке константе одређени су различити послекритични облици.*

Кључне речи: *Стабилност, еластична подлога, оптимални облик, послекритично понашање*