A NOTE ON MILANKOVITCH'S THEORY OF THRUST LINE APPLIED TO GOTHIC MASONRY ARCHES

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Summary: *Over the last few years, scholars have revisited the classical issue of identifying the limit equilibrium states of the symmetrical masonry arches of different shapes and corresponding theoretical minimum thickness, when subjected to self-weight. More than century ago, Serbian scholar Milutin Milankovitch in his remarkable work set the complete and correct theory of thrust line for the equilibrium of the arch of general shape, and computed the minimum thickness of semicircular arch. Although pointed arches, beside circular and elliptical, are very common in historic structures, particularly in Gothic architecture, the lack of information about their structural behaviour according to thrust line analysis is noticeable.*

In this paper, the characteristic elements of analysis, as well as computation, such as eccentricity, being the measure of pointedness, or the position of the application points of relevant forces, which diverse the pointed from semicircular arches, have been noticed. Furthermore, radial stereotomy, which assumes that joints between voussoirs are concurrent to the centre of the arch, considering both incomplete and overcomplete arches, is employed. Accordingly, the analytical expressions of the arch ring area and its centroid are provided, and, after equilibrium i.e. static approach, expression for the thrust line has been derived. Hence, it represents the basis for the various computational analyses of the mechanical behaviour of Gothic masonry arches, as well as other types of circular based arches and vaults containing the pointed crown.

Keywords: *pointed arches, equilibrium analysis, Milutin Milankovitch, thrust line theory*

1. INTRODUCTION

More than century ago, Serbian scholar Milutin Milankovitch in his remarkable, but for a long period nearly unknown work [1] (see [2] as well) set the complete and correct theory for the equilibrium of the masonry arch of general shape. He upgraded and exceeded preceding researchers, mainly architects, engineers and mathematicians, who, since the 18th century, in order to predict and prevent possible collapse of vaulted masonry structures, have developed various models applied in the stability and safety analysis (for more information see [3], [4]). Milankovitch considered the equilibrium of an arch on the assumptions about the material behaviour made from Couplet [5] onward: masonry has no tensile strength, it has infinite compressive strength and sliding cannot

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occur [3]. Furthermore, he followed the thrust line concept introduced in the 19th century (e.g. [6]), where thrust line (Milankovitch: *Druckkurven*) represents the load path, being the locus of the application points of the resultant thrust forces developed at the joints (beds) between the voussoirs of the arch. Milankovitch introduced in the computation the true location of the centre of the gravity of each ideal, generic voussoir, which was until then assumed to be located along the centreline of the arch. Hence, after remarkable mathematical elaboration, he was the first who gave the correct solution for the minimum thickness of semicircular arch of constant thickness subject to its own weight. In addition, he considered the different types of stereotomy, i.e. the direction of generic sections (joints), showing the multiplicity of limit thrust lines (see [7]).
Recently, few scholars have revisited the classical issue of identifying the limit equilibrium states of the symmetrical masonry arches of different shapes and corresponding theoretical minimum thickness, when subjected to self-weight (e.g. [8], [9]). Although pointed masonry arches, beside circular and elliptical, are very common in historic structures, particularly in Gothic architecture, their structural behaviour according to thrust line theory has not been researched in sufficient detail. However, in [10] the geometric framework is set, and the aim of the present paper is to provide a more detailed consideration on the line of thrust and its analytical expression. Hence, employing radial stereotomy (directions of the joints between voussoirs are concurrent to the centre of the arch, as used in Italian pointed arches), the characteristic elements of analysis, as well as computation, such as eccentricity, being the measure of pointedness, or the position of the application points of relevant forces, which diverse the pointed from semicircular arches, have been noticed, and are presented in the following section.

2. PARTICULAR GEOMETRIC PARAMETERS OF POINTED ARCH

Due to the symmetry of the arch, in the following discussion only half-arch is considered. In Fig. 1 (a) relevant geometrical parameters are shown: R and t denote the mean radius and the thickness of the arch ring, respectively. The minimum value of thickness to radius ratio, t/R, represents the minimum possible thickness of the arch normalized by the radius. Further, the value e, which measures the deviation from the circular shape, is the horizontal distance between the circular axis' centre O and the centre C of the pointed arch. The angle α represents the angle of embrace, which is the complement of the springing angle, and arches can be incomplete (segmental) or overcomplete (horseshoe), if this angle is less or greater than 90°, respectively. The substantial parameter of pointed arch is its eccentricity, being the measure of pointedness, and following [11] represents the ratio between e and the difference between R and e. Thus, arches can be slightly pointed (drop, depressed or obtuse arch), or strongly pointed (also known as lancet, acute or narrow pointed arch). The angle φ is angular coordinate measured from the vertical axis of the symmetry of the arch, which defines the generic section. Thrust line cannot be pointed, in the absence of concentrated loads [11], and therefore cannot pass through the extrados at the crown, as opposed to the circular arch. Hence, the parameters of particular importance are the application points B and S of the horizontal thrust H acting at the crown joint, and the reaction force R acting at the springings, since, among the eccentricity by default, their position affects the position of thrust line through the arch [10] and therewith the value of horizontal thrust, changing the location of critical sections as well.

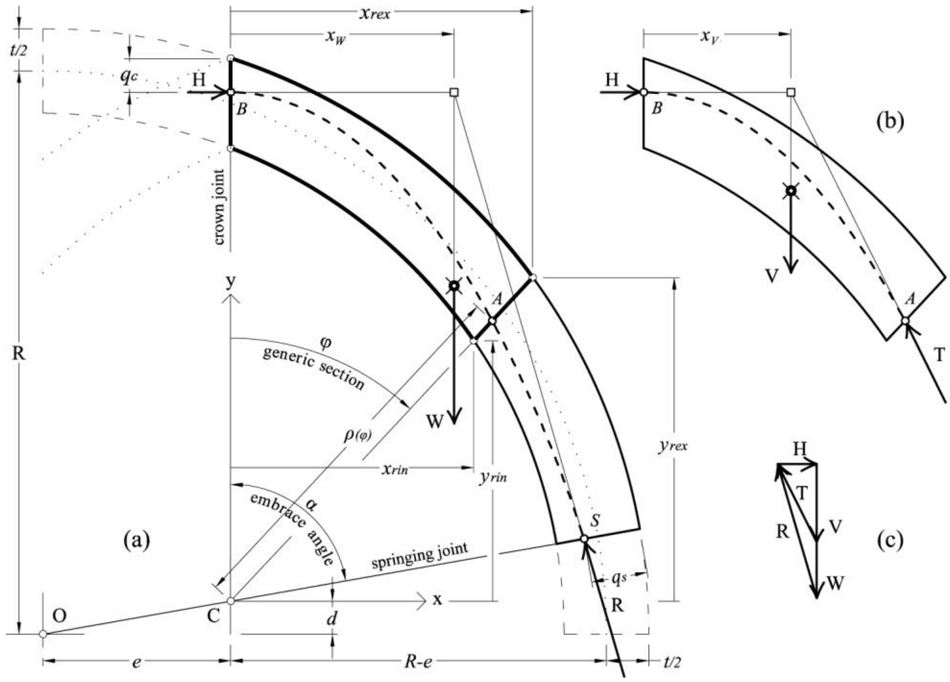

Figure 1. (a) Geometric parameters of pointed arch, (b) free-body diagram of the top portion of the arch with corresponding thrust forces acting on it, (c) force polygon [12]

The rigid arch is indeterminate to the third degree such that for any arch there is a family of possible equilibrium solutions, which can be visualized with lines of thrust obtained through graphical statics methods [11]. Accordingly, the force polygon expresses graphically the equilibrium of the system; the lines of action of the resultant thrust forces generate the funicular polygon, and the lines of action of the weights of the voussoirs meet at the corners of the funicular polygon to satisfy moment equilibrium (see [7]).

3. DERIVATION OF THE EXPRESSION FOR THE THRUST LINE THROUGH GOTHIC ARCHES

As pointed out by Milankovitch [1], the explicit equation of the line of thrust could be directly obtained without deducing the differential equation and then integrating it, when it is possible to find the analytical expression of the resultant load and its point of application for a finite portion of the system [2]. Thus, along with the usual assumption that specific weight of the masonry and the depth of the arch have unit value, for the sake of simplicity, the problem of the stability of arch can be reduced to purely geometrical task. Namely, a self-weight of the arch or its portion is substituted by the area of the arch ring, limited by extrados and intrados curves as well as by the particular joints between voussoirs, and is applied in the centre of gravity i.e. in the centroid of the limited area. Accordingly, knowing the weight *W* being the area of the half-arch as well as the centroid of the area, and assuming the application points *B* and *S* of the forces *H* and *F*, respectively (see Fig. 1 (a) and (c)), from rotational equilibrium about the springing hinge *S*, one can determine the value *H* of horizontal thrust, given by the following expression:

$$H = \frac{W\left(\rho_\alpha \sin(\alpha) - x_W\right)}{\rho_0 - \rho_\alpha \cos(\alpha)} \tag{1}$$

whereas the weight *W* of the half-arch, after [12], is:

$$W(\varphi=\alpha) = \frac{1}{2}\left[R_{ex}^2 \sin^{-1}\left(\frac{\sqrt{R_{ex}^2 - e^2}}{R_{ex}}\right) - e\sqrt{R_{ex}^2 - e^2} - R_{in}^2 \sin^{-1}\left(\frac{\sqrt{R_{in}^2 - e^2}}{R_{in}}\right) + e\sqrt{R_{in}^2 - e^2} - 2Rt\sin^{-1}\left(\cos(\alpha)\right)\right] \tag{2}$$

and the abscissa x_W of its centre of gravity is given by:

$$x_W(\varphi=\alpha) = \left\{ eR_{in}^2\left[\sin^{-1}\left(\frac{\sqrt{R_{in}^2 - e^2}}{R_{in}}\right) - \sin^{-1}\left(\cos(\alpha)\right)\right] - eR_{ex}^2\left[\sin^{-1}\left(\frac{\sqrt{R_{ex}^2 - e^2}}{R_{ex}}\right) - \sin^{-1}\left(\cos(\alpha)\right)\right] + \frac{1}{3}\left(e^2 - 2R_{ex}^2\right)\sqrt{R_{ex}^2 - e^2} - \frac{1}{3}\left(e^2 + 2R_{in}^2\right)\sqrt{R_{in}^2 - e^2} - \frac{2}{3}\cos(\alpha)\left(R_{ex}^3 - R_{in}^3\right) \right\} \Big/ \left[R_{ex}^2 \sin^{-1}\left(\frac{\sqrt{R_{ex}^2 - e^2}}{R_{ex}}\right) - e\sqrt{R_{ex}^2 - e^2} - R_{in}^2 \sin^{-1}\left(\frac{\sqrt{R_{in}^2 - e^2}}{R_{in}}\right) + e\sqrt{R_{in}^2 - e^2} - 2Rt\sin^{-1}\left(\cos(\alpha)\right)\right] \tag{3}$$

Furthermore, two boundary conditions defining the positions of application points *B* and *S*, i.e. the distance of the thrust line from the centre of the arch at the crown and springings, ρ_0 and ρ_α respectively, are as follows:

$$\rho_0 = \rho(\varphi=0) = \sqrt{\left(R + t/2\right)^2 - e^2} - d - q_c \tag{4}$$

and

$$\rho_\alpha = \rho(\varphi=\alpha) = \left(R + t/2\right) - q_s - \sqrt{d^2 - e^2} \tag{5}$$

whereas $d=e\cot(\alpha)$; q_c is the radial distance between the point of application of the horizontal thrust and the crown at extrados, and q_s is the radial distance between the point of application of the reaction force at the springings and the extrados, as shown in Fig. 1 (a). Further, the resultant thrust *T* at generic section at the angle φ together with its point of application *A* is uniquely determined from the force and moment equilibrium, either graphically with the force polygon (see Fig. 1 (b) and (c)) or analytically by solving equilibrium equations. Thus, from rotational equilibrium about point *A* follows:

$$V\left(\rho \sin(\varphi) - x_V\right) = H\left(\rho_0 - \rho\cos(\varphi)\right) \tag{6}$$

whereas, in accordance with Fig. 1 and after [12], the weight $V(\varphi)$ of the finite portion of the arch up to generic section at the angle φ, represented by the area of corresponding arch ring, and its centre of mass $x_V(\varphi)$ i.e. the centroid of that area, are given by Eq. (7) and Eq. (8), respectively:

$$\begin{aligned} V(\varphi) = \frac{1}{2}\Bigg\{ & (d+y_{rex})\left[2e - \sqrt{R_{ex}^2 - (d+y_{rex})^2}\right] - (d+y_{rin})\left[2e - \sqrt{R_{in}^2 - (d+y_{rin})^2}\right] - e\sqrt{R_{ex}^2 - e^2} + e\sqrt{R_{in}^2 - e^2} - \\ & R_{ex}^2\left[\sin^{-1}\left(\frac{d+y_{rex}}{R_{ex}}\right) - \sin^{-1}\left(\frac{\sqrt{R_{ex}^2 - e^2}}{R_{ex}}\right)\right] + R_{in}^2\left[\sin^{-1}\left(\frac{d+y_{rin}}{R_{in}}\right) - \sin^{-1}\left(\frac{\sqrt{R_{in}^2 - e^2}}{R_{in}}\right)\right] + \cot(\varphi)\left(x_{rex}^2 - x_{rin}^2\right)\Bigg\} \end{aligned} \tag{7}$$

$$\begin{aligned} x_V(\varphi) = \Bigg\{ & R_{ex}^2\left[e\sin^{-1}\left(\frac{d+y_{rex}}{R_{ex}}\right) + \sqrt{R_{ex}^2 - e^2} - e\sin^{-1}\left(\frac{\sqrt{R_{ex}^2 - e^2}}{R_{ex}}\right)\right] + (d+y_{rex})\left[e\sqrt{R_{ex}^2 - (d+y_{rex})^2} - \left(e^2 - R_{ex}^2\right)\right] - \\ & R_{in}^2\left[e\sin^{-1}\left(\frac{d+y_{rin}}{R_{in}}\right) + \sqrt{R_{in}^2 - e^2} - e\sin^{-1}\left(\frac{\sqrt{R_{in}^2 - e^2}}{R_{in}}\right)\right] + \frac{1}{3}\left[(d+y_{rex})^3 - \left(R_{ex}^2 - e^2\right)^{3/2} + \cot(\varphi)\left(x_{rex}^3 - x_{rin}^3\right)\right]\Bigg\} \Bigg/ \\ & \Bigg\{ (d+y_{rex})\left[2e - \sqrt{R_{ex}^2 - (d+y_{rex})^2}\right] - (d+y_{rin})\left[2e - \sqrt{R_{in}^2 - (d+y_{rin})^2}\right] - e\sqrt{R_{ex}^2 - e^2} + e\sqrt{R_{in}^2 - e^2} - \\ & R_{ex}^2\left[\sin^{-1}\left(\frac{d+y_{rex}}{R_{ex}}\right) - \sin^{-1}\left(\frac{\sqrt{R_{ex}^2 - e^2}}{R_{ex}}\right)\right] + R_{in}^2\left[\sin^{-1}\left(\frac{d+y_{rin}}{R_{in}}\right) - \sin^{-1}\left(\frac{\sqrt{R_{in}^2 - e^2}}{R_{in}}\right)\right] + \cot(\varphi)\left(x_{rex}^2 - x_{rin}^2\right)\Bigg\} \end{aligned} \tag{8}$$

In Eqs. (7) and (8), $R_{ex}=R+t/2$ and $R_{in}=R-t/2$ are radii of extrados and intrados circle, respectively, and the quantities x_{rex}, x_{rin}, y_{rex} and y_{rin} are the abscissas and ordinates of the extrados and intrados of the arch, as one can see in Fig. 1 (a). Finally, from Eq. (6) one can solve distance ρ between the thrust line and the centre of the arch, obtaining the following closed form expression for the thrust line of pointed arches (expressed in Cartesian coordinates and traced with dashed line in Fig. 1 (a) and (b)):

$$\rho(\varphi) = \frac{H\rho_0 + Vx_V}{H\cos(\varphi) + V\sin(\varphi)} \tag{9}$$

whereas the H, ρ_0, V and x_V are given by Eqs. (1), (4), (7) and (8), respectively (substituting these expressions into Eq. (9) one can obtain expanded expression for the line of thrust, not shown here because of the length of the solution). Hence, for each generic section, the distance of the thrust line, i.e. of the application point of the resultant thrust force, from the extrados and intrados, can be computed, and the critical sections regarding the joints where the thrust line approaches closest to the arch boundary can be detected.

4. FINAL REMARKS AND CONCLUSIONS

Serbian scholar Milutin Milankovitch was the first who provided the complete theory of thrust line, with the correct mathematical elaboration concerning the true location of the centres of gravity of generic voussoir, for the equilibrium analysis of the arch of general

shape. Recently, this theory has been revisited in the analysis regarding elliptical arches. However, the application to pointed arches has not been considered.
Therefore, in this paper, the characteristic elements of analysis, as well as computation, such as eccentricity, being the measure of pointedness or the position of the application points of relevant forces, which diverse the pointed from semicircular arches, have been noticed. Hence, after equilibrium i.e. static approach and employing radial stereotomy, concerning both incomplete and overcomplete pointed arches, the analytical expression for the thrust line has been derived. Therefore, it represents the basis for the various computational analyses of the mechanical behaviour of Gothic masonry arches, as well as other types of circular based arches or vaults containing the pointed part. Inspecting the different values of the eccentricity and embrace angle, critical sections with appropriate collapse modes can be identified, enabling the detection of the minimum thickness and corresponding thrust values. In addition, different types of exercised stereotomy, such as vertical or normal, remain to be considered. Moreover, regarding the further possible developments of the present research, appropriate iterative procedures and computer codes can be developed in order to enable the extensive elaboration of the problem as well as to provide the complete insight into the matter.

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БЕЛЕШКА О МИЛАНКОВИЋЕВОЈ ТЕОРИЈИ ПОТПОРНЕ ЛИНИЈЕ ПРИМЕЊЕНОЈ НА ГОТСКЕ МАСИВНЕ ЛУКОВЕ

Резиме: *Током последњих неколико година, научници поново обрађују класична питања одређивања стања граничне равнотеже симетричних масивних лукова различитих облика, те њихове одговарајуће минималне дебљине под сопственом тежином. Пре више од једног века, српски научник Милутин Миланковић је у свом изванредном раду поставио свеобухватну и исправну теорију потпорне линије намењену анализи равнотеже лука општег облика, те израчунао минималну дебљину полукружног лука. Иако су преломљени лукови, поред кружних и елиптичких, веома заступљени у историјским структурама, посебно у готској архитектури, приметан је недостатак информација о њиховом понашању према теорији потпорне линије.*

У овом раду су уочени карактеристични елементи анализе и прорачуна који разликују преломљене од полукружних лукова, као што су ексцентрицитет као мера преломљености лука, или положај нападних тачака релевантних сила. Осим тога, примењена је радијална стереотомија, којом се претпоставља да су лежишне спојнице између сводних каменова конкурентне с центром лука, узимајући у обзир и сегментне и потковичасте лукове. На основу тога су дати аналитички изрази за површину и тежиште прстена лука, те је према статичкој анализи изведен израз за потпорну линију. Тиме је постављена основа за различите рачунарске анализе механичког понашања готских масивних лукова, као и других кружних лукова и сводова који садрже преломљено теме.

Кључне речи: *преломљени лукови, статичка анализа, Милутин Миланковић, теорија потпорне линије*