

THE DESIGN OF TWO TYPES OF PLANE TRUSSES USING THE RELIABILITY INDEX

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Summary: In constant search for a more economical solution, engineering problems are threatened with tools of Statistics and Probabilistics, thus the contemporary design codes are usually based on the results of probabilistic analysis. In this paper we present a procedure for calculating the reliability index for two different timber trusses, which can serve as a basis for development of general structural design methods that include the Theory of probability, as well as an example of a probabilistic analysis procedure for those who need one.

Keywords: Reliability index, β index, plane truss, probabilistic design

1. INTRODUCTION

The search for more rational and economical solutions to engineering problems lead from deterministic concept of the allowed stresses, through the application of Statistics and Probabilistics to the concept of Limit States design of structures. Based on broad and comprehensive statistical data probabilistic models have been devised, but they were too complicated for an every-day use. So the results of probabilistic analysis were used to define partial safety factors and the probabilistic coefficients in the design codes (such as partial safety factor for material γ_M and Ψ factors in Eurocode, to name a few). Thus the design method used in the Codes is only "semi-probabilistic", and since the aim of this paper is to show the application of (fully) probabilistic methods, the analysed structures will not be designed by any of the design codes. Nevertheless, statistical data required to develop the probabilistic model will be extrapolated from the Eurocode 5 [1], that concerns itself with the design of timber structures.

In the first three sections some basic concepts and the used nomenclature will be presented, and the brief statement of the method for determining the reliability of structures will be given. In the next section the reliability index for two plane timber trusses will be calculated, followed by comparative analysis of the results and closing remarks and conclusions.

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2. ELEMENTS OF THE PROBABILITY THEORY

In this paper, **the probability** that an event A takes place will be denoted by

$$P(A) = n_A/n_{tot}$$

where n_A represents the number of experiments in which A occurs, and n_{tot} stands for the total number of experiments. The basic concepts and terms can be found in [2].

Here only the continuous random variables will be used and they'll be denoted with capital letters and the values they can take will be denoted with small letters, for example X and x .

Cumulative distribution function (CDF) of the random variable X will be presented as the probability that X takes on a value that is less than x : $F_X(x) = P(X < x)$.

Probability density function (PDF) is then the probability that X takes a value in the infinitesimal neighbourhood of value x : $f_X(x) = P(X \in [x, x + dx]) = dF_X(x)/dx$.

Therefore, the next equality holds:

$$F_X(x) = \int_{-\infty}^x f_X(x) dx$$

Distribution functions can be defined by their parameters or by their moments. In this paper, only the first two moments, corresponding to the first two parameters (**mean value μ_X and standard deviation σ_X**) will be used, and the relationship between them is given by:

$$\mu_X \equiv \bar{x} = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$\sigma_X^2 \equiv s^2 = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx$$

Here we have restricted ourselves to using only the **normal distribution function (NDF)** and the **lognormal distribution function (LNDF)**. NDF of the variable X with mean value μ_X and standard deviation σ_X ($n(x; \mu_X, \sigma_X)$) and LNDF of the variable X with mean value μ_X and standard deviation σ_X ($\log n(x; \mu_X, \sigma_X)$) are given by:

$$n(x; \mu_X, \sigma_X) = \frac{1}{\sigma_X \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x - \mu_X}{\sigma_X} \right)^2}$$

$$\log n(x; \mu_X, \sigma_X) = n(\ln x; \mu_X, \sigma_X) = \frac{1}{\sigma_X \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{\ln x - \mu_X}{\sigma_X} \right)^2}$$

Cumulative distribution function of $n(x; \mu_X, \sigma_X)$ (CDFN) can then be expressed as:

$$N_X(x; \mu_X, \sigma_X) = \int_{-\infty}^x n(x; \mu_X, \sigma_X) dx$$

3. RELIABILITY OF STRUCTURES

If a construction comes into any unacceptable or in any way undesirable state, it is said that the failure of construction has occurred. Whether the failure occurs or not depends on a lot of factors that can, roughly, be divided into two groups: external, that is actions of loads and actions on structures in general, and internal, such as material properties and geometrical characteristics of elements. If we denote actions on structure with s , and

corresponding resistances of structure with r , the failure occurs when $r \leq s$. To take uncertainties into account, at least one of the variables has to be considered a random variable. If we let the resistances vary, we can define the **probability of failure** P_f :

$$P_f = P(R \leq s) = F_R(s) = \int_{-\infty}^s f_R(x) dx$$

Most often, it is necessary to consider the actions and loads as undeterministic, described with the random variable S , and the probability of failure is then given by:

$$P_f = P(R \leq S) = P(R - S \leq 0) = \iint_{D_f} f_{RS}(r, s) dr ds$$

where $f_{RS}(r, s)$ is the joint PDF of random variables R and S , and D_f is the part of the domain of f_{RS} in which the failure occurs (the failure domain). However, in general case both actions and resistances are functions of several random variables. If all the random variables considered are organised into a random vector \mathbf{X} , actions and resistances become $R(\mathbf{X})$ i $S(\mathbf{X})$. Then it is more convenient to introduce the **state function** $g(\mathbf{X})$ defined as $g(\mathbf{X}) = R(\mathbf{X}) - S(\mathbf{X})$. Now the n -dimensional space of the state function can be divided in two regions: failure domain, where $g(\mathbf{X}) \leq 0$, and safe region, where $g(\mathbf{X}) > 0$. The boundary between them is called the **limit state function** and it can, in general, be (and almost always is) nonlinear. The probability of failure is then calculated as:

$$P_f = P(g(\mathbf{X}) \leq 0) = \int_{g(\mathbf{X}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) dx_1 \dots dx_n$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint PDF for all the random variables.

The solution to this integral in the closed form does not exist, so some of the numerical methods are usually applied to solve it. Here we will use the Hasofer-Lind method, basics of which will be displayed shortly.

It is important to introduce the concept of **reliability index** β . For elementary case where both actions and resistances are the functions of only one random variable with normal distribution, the limit state function is linear function of also one random variable with normal distribution, and it is called *the margin function* Z :

$$\begin{aligned} f_R(x) &= n(\mu_R, \sigma_R), \quad f_S(x) = n(\mu_S, \sigma_S) \\ Z = g(x) &= R - S \\ f_Z(x) &= n(\mu_Z, \sigma_Z) \end{aligned}$$

Obviously, the probability of failure is then

$$P_f = P(Z \leq 0) = \int_{-\infty}^0 f_Z(x) dx = \int_{-\infty}^0 n(\mu_Z, \sigma_Z) dx = N(0; \mu_Z, \sigma_Z)$$

Expressed in terms of standardised CDFN that we'll denote $\Phi(x) \equiv N(x; 0, 1)$, it holds:

$$P_f = P(Z \leq 0) = N(0; \mu_Z, \sigma_Z) = \Phi\left(\frac{0 - \mu_Z}{\sigma_Z}\right) = \Phi\left(-\frac{\mu_Z}{\sigma_Z}\right) = \Phi(-\beta)$$

where $\beta = \frac{\mu_Z}{\sigma_Z}$ is called *the reliability index* and it gives an idea of the reliability of the structure.

However, if the limit state function (LSF) is a nonlinear function or a function of multiple or non-normally distributed random variables, it is much more complicated to

determine the β index. If the LSF is developed in Taylor's series in the design point (it represents the most likely combination of values of all the random variables that would lead to a failure) and only the linear terms are kept while the terms of the higher order are neglected, that is the FORM analysis - First Order Reliability Method. If only the first two moments of the random variables' distribution functions are taken into account, it is FOSM method - First Order Second Moment method. This method has a serious disadvantage - it gives different values for β index depending on different but equivalent mathematical formulations of the same mechanical problem. That is why AFOSM methods have been developed - Advanced FOSM methods, one of which is the Hasofer-Lind method which is used in this paper.

The Hasofer-Lind method is based on standardization of the random variables. Here we cannot adduce the algorithm for this method due to the lack of space, but the interested reader is referred to [3] where he or she can find it described in detail.

Using the mentioned algorithm reliability index of any structural element can be calculated, given that the parameters of the random variables are known. There are several computer programmes created for this purpose, and for this paper we've used the Free VaP 1.6 software.

Once the β index is known, probability of failure is readily calculated as $P_f = \Phi(-\beta)$ and the probability of survival of the construction is then:

$$P_s = \int_s^{\infty} f_R(x) dx = \int_{-\infty}^{\infty} f_R(x) dx - \int_{-\infty}^s f_R(x) dx = 1 - P_f$$

4. RELIABILITY OF STRUCTURAL SYSTEMS

So far we've showed how to determine the reliability index of an individual element, but structures are almost exclusively systems of elements and the global reliability index for a structure as a whole is calculated somewhat differently, as follows. (For more details consult [4].)

First we determine the **construction failure model** by assembling the **failure components** in the appropriate way. *Failure component* is every one of the considered failure cases, and these components can be linked together in series or parallel or in combination of these two. For example, if we analyse a statically determined plane truss with n bars, m of which are in compression, for every bar we can formulate a failure criterion and the corresponding LSF, and we can also formulate an additional failure condition for buckling of every compressed truss member. This would result in $m+n$ failure components, and we should link them in series because if any one of the failure states is realised, the whole truss collapses. In this paper we will consider only these types of trusses.

So, as previously mentioned, for n structural elements we can formulate m failure elements (where $m \geq n$), the latter being LSFs $g_i(\mathbf{X})$, $i = 1, 2, \dots, m$. For every failure element there is a probability of failure $P_{f_i} = P(g_i(\mathbf{X}) \leq 0)$ and the probability of failure of the system can then, in terms of AFOSM analysis, be determined by

$$P_f^S \approx 1 - \Phi_m(\beta; \rho) = \Phi(-\beta^S)$$

where β is the vector of all the individual reliability indices, ρ is the matrix of the directional cosines of the outward normals to the LSFs, Φ is one-dimensional NDF, β^S is the reliability index of the whole system and Φ_m is a m -dimensional NDF representing the integral:

$$\Phi_m = \int_{-\infty}^{\beta_1} \int_{-\infty}^{\beta_2} \dots \int_{-\infty}^{\beta_m} \varphi_m(\mathbf{x}; \rho) dx_1 dx_2 \dots dx_m$$

where $\varphi_m(\mathbf{x}; \rho)$ stands for the m -dimensional PDFN.

Solving this integral is a formidable task so it is seldomly done. Instead, we use the method of bounds to determine between which bounds the real value of β^S is. There are several bounds methods, and we will use *the method of simple bounds*. It gives somewhat broad span of values but it is rather easy to use and accurate enough for the purposes of this paper. According to this method [4], the reliability index for the system finds itself between the following bounds:

$$\min_{i=1,2,\dots,m} \beta_i \geq \beta^S \geq -\Phi^{-1}\left(\sum_{i=1}^m P_{f_i}\right) \quad (1)$$

where Φ^{-1} is an inverse function of Φ .

5. CALCULATION OF THE RELIABILITY INDICES OF THE TWO PLANE TRUSSES

In this paper we will examine two solutions to the construction problem of a plane timber truss. One truss will have diagonals in tension, and the other will have diagonals in compression. We will iteratively design the trusses using the reliability indices as the criterion, and afterwards evaluate both solutions from the aspect of reliability. The geometry and the loading scheme are shown in Figure 1.

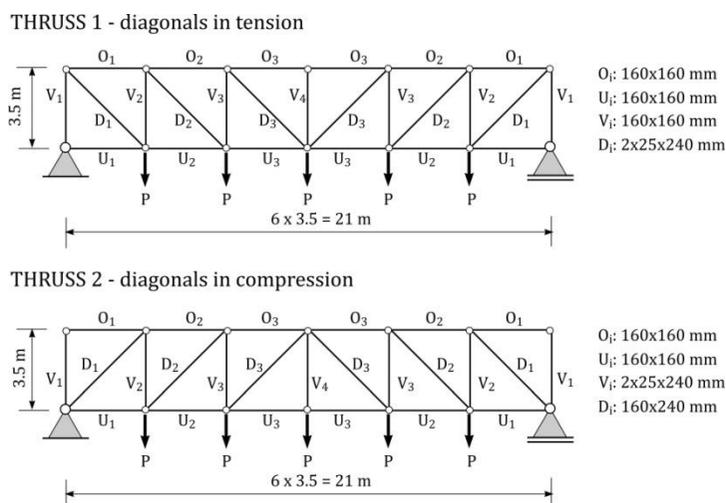


Figure 1. The geometry and the loading scheme

The Eurocode [5] specifies that the reliability index for a structural system should not be less than 3.50 for the ultimate limit state, and 1.50 for the serviceability limit state, and these regulations were used to determine the required dimensions of structural elements, through an iterative process. The dimensions of members given in Figure 1 are the final dimensions that satisfy all the considered design conditions. Please note that some of the members have complex cross-sections for constructional reasons. While the geometry of the trusses is treated as deterministic, the material properties as well as the loads are modelled probabilistically and are introduced as the random variables.

For the purpose of this paper the loading factor P will be considered as a random variable with normal distribution with the mean value of 40 kN and standard deviation of 4 kN. Thus we've assumed the loads pretty arbitrarily, but for material properties we use the data that can be found in the Eurocodes. The applied material for both trusses will be sawn wood of the class C30, and its characteristics are given in Table 1. taken from Eurocode 5 [6]:

Table 1: Material properties

Timber class	Tensile strength parallel to fibers $f_{t,0,k}$ [N/mm ²]	Compressive strength parallel to fibers $f_{c,0,k}$ [N/mm ²]	Elasticity modulus [N/mm ²]		Specific mass (density) ρ_k [kg/m ³]
			$E_{0,mean}$	$E_{0,0.05}$	
C30	18	23	13 000	8 000	380

The values for $f_{t,0,k}$, $f_{c,0,k}$ and $E_{0,0.05}$ in Table 1. are *characteristic* values, meaning that they correspond to the 0.05% fractile of the used distribution function for the considered property. Since we want to use the fully probabilistic approach, we will need to extrapolate the parameters for the corresponding distribution function and then use that function to calculate the reliability index. Since the Eurocode uses lognormal distribution for material properties, if we assume the value for the coefficient of variation $v = 10\%$ (the ratio between the mean value and the standard deviation), we can determine the mean value and the standard deviation for material strengths from the following two systems of equations:

$$\int_{-\infty}^{f_{t,0,k}} \log n(\mu_{f_t}, \sigma_{f_t}) dx = 0.05$$

$$\sigma_{f_t} = v_{f_t} \mu_{f_t} = 0.1 \mu_{f_t}$$

$$\int_{-\infty}^{f_{c,0,k}} \log n(\mu_{f_c}, \sigma_{f_c}) dx = 0.05$$

$$\sigma_{f_c} = v_{f_c} \mu_{f_c} = 0.1 \mu_{f_c}$$

This way we get the values for the first two parameters of the distribution functions for material tensile and compressive strength parallel to fibres:

$$\mu_{f_t} = 21.32 \text{ N/mm}^2$$

$$\sigma_{f_t} = 2.13 \text{ N/mm}^2$$

$$\mu_{f_c} = 27.24 \text{ N/mm}^2$$

$$\sigma_{f_c} = 2.72 \text{ N/mm}^2$$

For the elasticity modulus we already have the characteristic and the mean value given in Table 1. Then, for the assumed coefficient of variation of $v = 12.5\%$ we can calculate the standard deviation:

$$\int_{-\infty}^{E_{0,0.05}} \log n(\mu_E, \sigma_E) dx = 0.05 \Rightarrow \sigma_E = 3610.42 \text{ N/mm}^2$$

For convenience, the parameters for all the random variables are recapitulated in Table 2

Table 2: Parameters of the distribution functions of the considered random variables

	f_t [N/mm ²]	f_c [N/mm ²]	E [N/mm ²]	P [N]
μ	21.32	27.24	13000.00	40 000.00
σ	2.13	2.72	3610.42	4 000.00

For each of the failure cases we can define the corresponding limit state function. The truss will collapse if the stresses in any member exceed the material strength, or if it comes to the buckling of any of the compressed members, i.e. the axial force in a compressed member is greater than or equal to the critical buckling load (here we limit ourselves only to linear buckling analysis). Therefore, bearing in mind that the random variables are material tensile strength ($f_t \equiv X_1$), the loading factor ($P \equiv X_2$), material compression strength ($f_c \equiv X_3$) and the modulus of elasticity ($E \equiv X_4$), the limit state functions are:

$$g_i^{[1]} = f_t - \sigma_i = f_t - \frac{N_i}{A_i} = f_t - \frac{k_{P_i}}{A_i} P = X_1 - \frac{k_{P_i}}{A_i} X_2$$

$$g_i^{[2]} = f_c - \sigma_i = f_c - \frac{N_i}{A_i} = f_c - \frac{k_{P_i}}{A_i} P = X_3 - \frac{k_{P_i}}{A_i} X_2$$

$$g_i^{[a]} = P_{cr} - N_i = EI \left(\frac{\pi}{l_{0_i}} \right)^2 - k_{P_i} P = I_i \left(\frac{\pi}{l_{0_i}} \right)^2 X_4 - k_{P_i} X_2$$

where $g_i^{[1]}$ is valid for any member in tension, and $g_i^{[2]}$ and $g_i^{[a]}$ are valid for any member in compression. We use σ_i for normal stress, N_i for axial force, A_i for cross-sectional area, I_i for moment of inertia of the cross-section, and l_{0_i} for the free buckling length of the i -th member. We also introduce the member axial force coefficient k_{P_i} , representing the axial force in the i -th member due to the unit load factor P. The values for k_{P_i} for both analysed trusses are shown in the Figure 2.

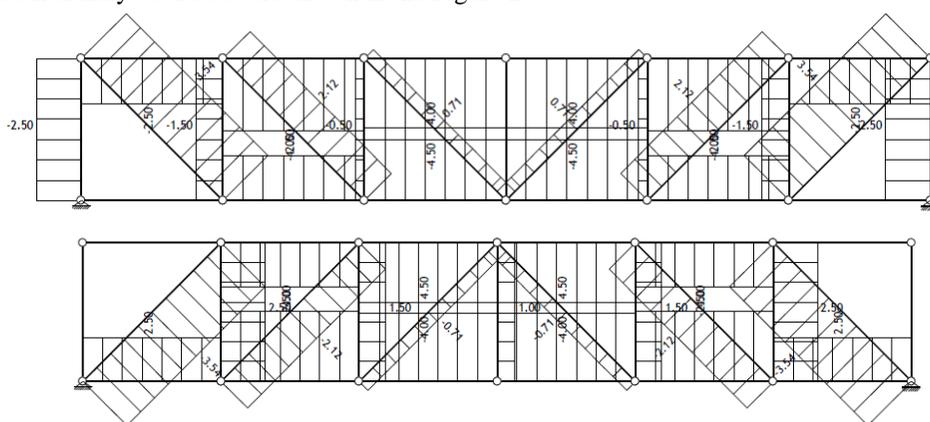


Figure 2 - Axial forces due to the load $P=1.0$

Using the described Hasofer-Lind algorithm we have calculated the individual reliability index for each member of both trusses in the Free VaP 1.6 software, and the results are given in Table 3.

TRUSS 1		
Member	Reliability index	Probability of failure
U1	50.00	0.00E+00
U2	13.60	2.00E-42
U3	9.61	3.63E-22
D1	4.41	5.17E-06
D2	8.55	6.15E-18
D3	18.00	9.74E-73
O1	15.70	7.56E-56
O2	11.70	6.37E-32
O3	10.70	5.09E-27
V1	15.70	7.56E-56
V2	20.20	4.89E-91
V3	30.20	1.18E-200
V4	50.00	0.00E+00
Buckling analysis		
O1	5.93	1.51E-09
O2	4.29	8.93E-06
O3	3.88	5.22E-05
V1	5.93	1.51E-09
V2	7.73	5.38E-15
V3	11.60	2.06E-31
V4	50.00	0.00E+00

TRUSS 2		
Member	Reliability index	Probability of failure
U1	13.60	2.00E-42
U2	9.61	3.63E-22
U3	50.00	0.00E+00
D1	16.20	2.52E-59
D2	20.80	2.16E-96
D3	30.70	2.85E-207
O1	50.00	0.00E+00
O2	15.70	7.56E-56
O3	11.70	6.37E-32
V1	50.00	0.00E+00
V2	7.22	2.60E-13
V3	11.50	6.60E-31
V4	15.00	3.67E-51
Buckling analysis		
O1	5.93	1.51E-09
O2	4.29	8.93E-06
O3	3.88	5.22E-05
D1	3.71	1.04E-04
D2	5.50	1.90E-08
D3	9.35	4.38E-21

Since the trusses are statically determinate, failure components are linked in series and then the reliability indices for the trusses as a whole were calculated by Equation 1, giving:

$$3.88 \geq \beta^S \geq 3.65 \quad \text{and} \quad 3.71 \geq \beta^S \geq 3.41$$

However, all of these calculations were made for the Ultimate Limit State (ULS), and the failure occurs not only if the construction collapses, but also when it does no longer meet the serviceability requirements. Thus we need to analyse the Serviceability Limit State (SLS) as well. Let us assume that for the problem at hand the allowed deflection is $w_{ul} = L/k_w = L/300$. The maximum deflection of the truss can be calculated by applying the Principal of Virtual Forces. Axial forces in members due to the unit dummy load P at the centre of the span are shown in Figure 3.

The limit state function corresponding to the SLS can then be expressed as:

$$g^{[4]} = w_{ul} - w_{max} = \frac{l}{k_w} - \int \frac{N\bar{N}}{EA} ds = \frac{l}{k_w} - \sum \frac{N_i \bar{N}_i l_i}{EA_i} = \frac{l}{k_w} - \frac{P}{E} \sum \frac{k_{P_i} \bar{N}_i l_i}{A_i} = \frac{l}{k_w} - \frac{P}{E} \sum \alpha_i$$

However, we will transform the LSF $g^{[4]} = 0$ to a form that is more convenient to use:

$$g^{[4]} = \frac{l}{k_w} - \frac{P}{E} \sum \alpha_i \leq 0 \quad \rightarrow \quad g^{[4]} = lE - k_w P \sum \alpha_i = lX_4 - \left(k_w \sum \alpha_i \right) X_2 \leq 0$$

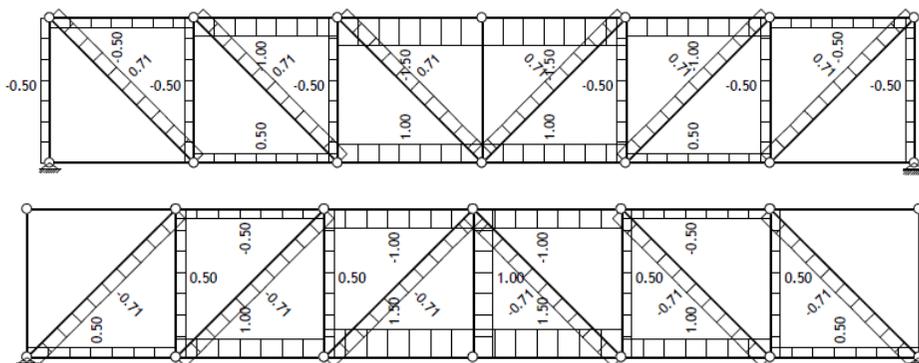


Figure 3 - Axial forces due to the unit dummy load at the center of the span

Using the Hasofer-Lind method to calculate the SLS reliability indices for trusses, we obtain:

$$\beta_w = 3.06 \quad \text{and} \quad \beta_w = 6.22$$

6. COMPARATIVE ANALYSIS OF THE RESULTS

The results show that both trusses satisfy the proposed criterion for reliability index: for ULS, Truss 1 has an average reliability index of $\beta_{ULS}^{tens.} = 3.77$, which is by 8% higher than the required value of 3.50, and Truss 2 has an average reliability index of $\beta_{ULS}^{compr.} = 3.56$ that is by 2% higher than required. For SLS, Truss 1 has $\beta_{SLS}^{tens.} = 3.06$ that is about 2 times higher than the required value of 1.50, and Truss 2 has $\beta_{SLS}^{compr.} = 6.22$ that is more than 3 times greater than required. There we can see that the solution with diagonals in tension is more reliable in terms of survival of construction, and that it is at the same time closer to the required serviceability state limit, leading to less over-designed structure compared to the other solution. If we add to that the fact that less material is used for the Truss 1 (it requires 2.06m³ compared to 2.51m³ required for the Truss 2, giving the difference of more than 20%), it can be concluded that the constructional solution of a timber plane truss with diagonals in tension is better than the one with diagonals in compression.

7. REMARKS AND CONCLUSIONS

Of course, the conclusions made here are not universally valid, since we have analysed only two types of trusses and have restricted our investigation only to timber structures. However, this paper shows how the reliability concepts and the Theory of probability can be used in the structural design process directly, rather than through the stipulated values for various factors and coefficients as they are currently included in the structural design codes.

In authors' opinion, it is very important that the deterministic approach to engineering problems be abandoned, for at almost all the relevant cases the nature of the problem is too complex or us to describe it so precisely that we can derive conclusions with absolute certainty. Thus, until we have found the fundamental relations underlying the considered problem, we need to rely on statistics and probability in order not to over-design our structures (too much), and the reliability analysis is a perfect tool to achieve this. That is why one of our future tasks should be to develop an universal structural design software based on the reliability concept, that would introduce uncertainties and handle them successfully, rather than ignore them.

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ДИМЕНЗИОНИСАЊЕ ДВА ТИПА РАВНИХ РЕШЕТКИ ПРЕМА ИНДЕКСУ ПОУЗДАНОСТИ

Резиме: У циљу економичности, инжењерски проблеми се данас сагледавају кроз Статистику и Теорију вероватноће на су и грађевински прописи (какав је и Еврокод) углавном засновани на резултатима пробабилистичке анализе. У овом раду је приказан поступак одређивања индекса поузданости на примеру две различите дрвене решетке, што може послужити као полазна основа за развој опитних метода димензионисања конструкција према Теорији вероватноће, и као преглед поступка пробабилистичке анализе за оне који тек залазе у ову област.

Кључне речи: Индекс поузданости, решетке, пробабилистичко димензионисање