NUMERICAL ANALYSIS OF ROD ACCORDING TO THE LARGE DISPLACEMENT THEORY

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Summary: *The paper presents a procedure for numerical modelling of the geometric nonlinearity of a rod. The calculation of cross-sectional forces, displacements and rotations of nodes was done by iterative methods on a deformed system. By the described procedure, the equilibrium state is established in the finite position of the rod. In the process of deformation, there is an increase in cross-sectional forces and deformation of the rod. The presented calculation methods are used to model geometric nonlinearity with constant and variable stiffness of the cross section of the rod. The calculations were done numerically, and the results were controlled using the SCIA software package. Through numerical examples, the calculation procedure was presented and the analysis of the results was performed.*

Keywords: *The large displacement theory, geometric nonlinearity of a rod*

1. INTRODUCTION

Static and deformation sizes of the axis of the rod are determined from three types of equations. The equations describe the static and geometric connections, and the material properties of the rod. In a significant number of cases of calculation of line beams, assumptions about geometric and material linearity are introduced into the equations. Assumptions about geometric linearity are based on small values of the deformation along the axis of the rod ε, rotation φ and elementary displacements of rod axis elements du and dv. The material linearity of the problem implies a linear relationship between stresses and strain sizes of the rod [1]. In the large displacement theory of the rod axis element, the assumption of material linearity is retained. The geometric nonlinearity of the rod is introduced into the calculation. The analysis of beams with slender elements in which large (finite) deformations of rods occur is justified according to the theory of finite deformations. The displacement of the rod axis at large deformations consists of a horizontal and a vertical component. In the paper [2], the solution of deformations in

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cantilever beam is presented using Lagrange formulation. Kinematic connections have been established for the Euler-Bernoulli beam between the displacements of the nodes at finite deformations. Then the displacement of the rod axis is interpolated by a polynomial passing through three nodal points of the rod. The calculation of the finite deformations of the beam by energy methods is presented in the paper [3]. The deformation energy of two orthogonal beams in the paper [3] was determined by introducing squares and higher degrees of deformation ε of the rod axis. The stiffness matrices of the system are determined for the adopted functions of horizontal and vertical displacement of the rod axes. The procedure for solving the beam bending problem by the finite deformation method is presented using a software algorithm. In the paper [4], the cantilever deflection problem is described by the nonlinear differential equation of beam element bending at finite deformations. The solutions of the differential equation show the analytical expressions for the displacements of the cantilever rod.

2. ANALYSIS OF THE ROD ELEMENT FOR LARGE DISPLACEMENT

By analysing the deformation quantities and cross-sectional forces of the rod on the deformed axis, we obtain the large displacements, rotations and forces of the rod axis. The material behaviour is assumed to be linearly elastic. By introducing geometric nonlinearity, the force-displacement and deformation-displacement relations are nonlinear. The rod theory based on these relations is called the large displacement theory or the third-order theory [8]. The theory of large displacement defines the problem of geometric nonlinearity of the system. Therefore, the equations describing the geometric connections of the system and the equilibrium conditions are nonlinear. In the deformation equations, the following holds: $\varepsilon \neq 0$, $\varphi \neq 0$, $du \neq 0$, $dv \neq 0$. The problem of bending the elementary part of a straight rod is shown in Figure 1. Taking into account the elements from Figure 1.a [6], the length of the element *ds* after deformation can be determined in the form:

$$ds = du + \sqrt{dx^2 + (\varphi \cdot dx)^2} = du + \sqrt{dx^2 + \left(\frac{dv}{dx}\right)^2 dx^2} = du + dx\left[1 + \left(\frac{dv}{dx}\right)^2\right]^{\frac{1}{2}} \tag{1}$$

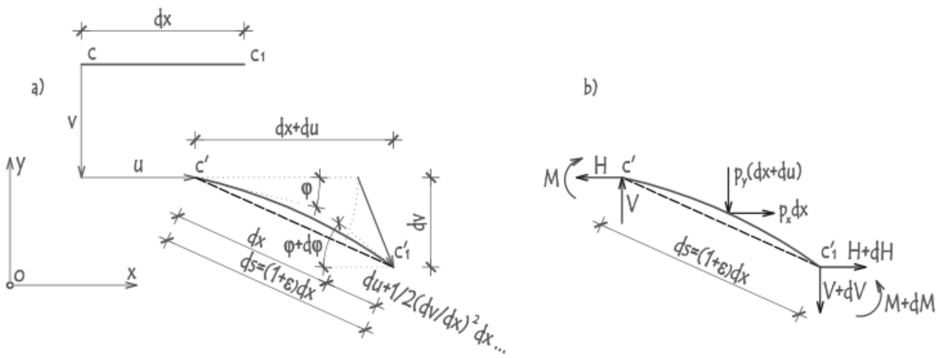

Figure 1. a) Displacements of the axis of the rod element, b) forces on the rod element

The member in brackets develops into a binomial order of the form:

$$\left[1+\left(\frac{dv}{dx}\right)^2\right]^{\frac{1}{2}}=1+\frac{1}{2}\left(\frac{dv}{dx}\right)^2-\frac{1}{8}\left(\frac{dv}{dx}\right)^4+... \quad (2)$$

After replacing expression (2) in (1), the finite shape for the length of the rod element after deformation is:

$$ds=dx+du+\frac{1}{2}\left(\frac{dv}{dx}\right)^2dx-\frac{1}{8}\left(\frac{dv}{dx}\right)^4dx+... \quad (3)$$

For a straight rod without taking into consideration the shear deformation fraction $\gamma = 0$, the equations describing the static and deformation states of the rod are of the form [8]:

$$\begin{aligned}
&dH+p_x\,dx=0 &\qquad& dx+du=(1+\varepsilon)dx\cdot\cos\varphi\\
&dV+p_y\,dx=0 && dv=(1+\varepsilon)dx\cdot\sin\varphi\\
&dM-V(dx+du)+H\,dv=0 &&\\
&\varepsilon=\frac{1}{EA}(H\cos\varphi+V\sin\varphi)+\alpha_t\cdot t_0 &&\\
&\kappa=\frac{d\varphi}{ds}=-\frac{M}{EI}-\alpha_t\frac{\Delta t}{h} &&
\end{aligned} \quad (4)$$

The system of equations (4) consists of seven nonlinear equations, with seven unknown quantities: H, V, M, u, v, φ, and ε in which elements of unknown static and deformation quantities occur. It is convenient to solve nonlinear equations in a numerical way, which implies carrying out an iterative procedure to a satisfactory solution [1]. In each iterative step, there is a change in the cross-sectional forces and deformations of the rod. Therefore, the solution is sought in several steps (iteration) until the deformation difference between two consecutive steps is small enough. Certain assumptions and limitations have been introduced in the calculation, such as [10]:

- The connections between stress and deformation are linear;
- The influence of transversal forces is neglected;
- The geometrical characteristics of the cross section of the rods before and after deformation are the same.

3. NUMERICAL SOLUTION OF ROD DISPLACEMENT

3.1 Case $(dv/dx)^2<1$

The numerical solution of bending of a rod by the method of the large displacement theory is based on the addition of higher order members to the deformations of the rod and the iterative procedure. If we apply a load to a straight, statically determined rod with EI=const., there is a rotation and translational displacements of the points of that rod. By dividing the rod into smaller elements, unknown deformations are at the ends of the elements. Due to the movement of the ends of the elements, an angle ψ_{ik} is formed, which represents the angle of rotation of the axis of the rod element, Figure 2 [10].

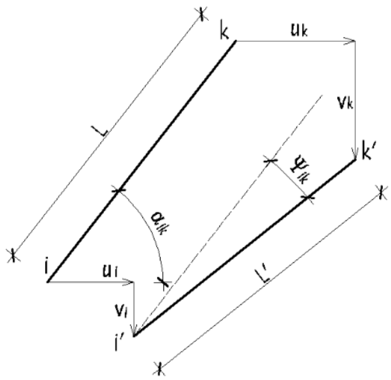

Figure 2. Rotation of the rod axis chord due to displacement of the ends

According to Figure 2, when the relative deformation is $\varepsilon \neq 0$, the length of the rod changes. For different displacements of the ends of the rod *i-k*, the angle of rotation of the axis of the rod is formed, which from Figure 2 can be expressed in the form:

$$\mathrm{tg}\psi_{ik} = \frac{u_k - u_i}{L'}\sin\alpha_{ik} + \frac{v_k - v_i}{L'}\cos\alpha_{ik} \tag{5}$$

Neglecting the shear deformations, the displacements *v* and the rotation φ of the rod points are affected by the relative deformation ε and the curvature of the rod κ. The relative deformation ε is determined as the ratio of the change in the length of the rod element and the initial length dx. The change in the length of the rod element ds-dx is obtained from expression (3) in the form:

$$ds - dx = du + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx - \frac{1}{8}\left(\frac{dv}{dx}\right)^4 dx + \ldots \tag{6}$$

If we take into account three members of the order, the expression for the relative deformation ε of the rod axis in the theory of large displacement is in the form:

$$\frac{ds - dx}{dx} = \varepsilon = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 - \frac{1}{8}\left(\frac{dv}{dx}\right)^4 \tag{7}$$

where the terms $(1/2)\cdot(dv/dx)^2 - (1/8)\cdot(dv/dx)^4$ are higher order elements that represent the change in relative deformation due to the displacement of the rod axis. In the case of small displacements, it is sufficient to consider the first member of the order. The new rod length is as follows:

$$L' = L(1+\varepsilon) \tag{8}$$

The change in the longitudinal displacement of the rod axis is calculated on the basis of expression (7) in the form:

$$\frac{du}{dx} = \varepsilon - \frac{1}{2}\left(\frac{dv}{dx}\right)^2 \Rightarrow \Delta u = \int_0^L \varepsilon\, dx - \frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx \tag{9}$$

During the process of deformation of the beam, the cross-sectional forces and deformations change. The shear forces are determined from the conditions of rod equilibrium, and the displacements are determined from the conditions of equality of the virtual work of external and internal forces. According to this condition, the equilibrium state of the system occurs when the works of virtual forces are equalized in the form:

$$\overline{P}\cdot\Delta v=\int_0^s \overline{M}\cdot\Delta\kappa\, ds+\int_0^s \overline{N}\cdot\Delta\varepsilon\, ds \tag{10}$$

For the local axis of the rod ($ds=dx/\cos\psi_{ik}$) at the finite deformation by replacing the increment of the curve of the rod $\Delta\kappa$, the relative deformation $\Delta\varepsilon$, for $\overline{P}=1$ the expression for the displacement increment is obtained as following:

$$\Delta v=\int_0^x \overline{M}\cdot\frac{\Delta M}{EI\cdot\cos\psi_{ik}}dx+\int_0^x \overline{N}\cdot\frac{\Delta N}{EA\cdot\cos\psi_{ik}}dx \tag{11}$$

where $\overline{M}, \overline{N}$ are the virtual cross-sectional forces from the unit load, and ΔM and ΔN are the bending moments increments and normal forces from the external load.

3.2 Case $(dv/dx)^2>1$

The development of the member $[1+(dv/dx)^2]^{1/2}$ into a binomial order is not possible for the case when $(dv/dx)^2>1$. When angle of rotation of the rod axis Ψ_{ik} is greater than 45^0, the order of shape (2) is no longer valid. The solution for moving the rod axis is sought iteratively, based on the kinematic connections of the deformed rod according to Figure 3. In Figure 3a. the displacement of the node *i* is zero, and the finite displacement of the node *k* causes the rotation of the rod *i-k*. In this case, the node *i* represents the pole of rotation of the chard of the rod *i-k*.

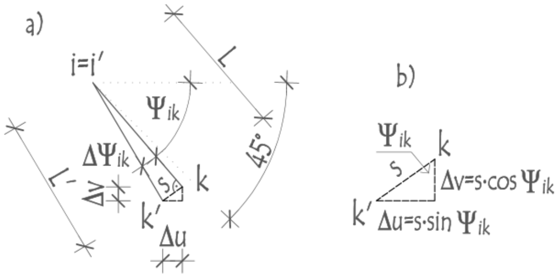

Figure 3. a) Displacement of the rod chard, b) increment of displacement of the node k

If the increment of vertical displacements Δv of the node *k* of the rod is small, then the chord s=k-k' is approximately equal to the length of the arc described by the axis of the rod when rotating around the node *i*, so it follows:

$$s=tg\,\Delta\psi_{ik}\cdot L \tag{12}$$

According to the geometric relations in Figure 3.b, the increment of the horizontal displacement Δu of the node *k* is of the form:

$$\Delta u=tg\,\Delta\psi_{ik}\cdot L\cdot\sin\psi_{ik} \tag{13}$$

The increment of the angle $\Delta\Psi_{ik}$ is determined by the expression:

$$tg\Delta\psi_{ik}=\frac{(v_{k'}-v_k)-(v_{i'}-v_i)}{x_k-x_i} \tag{14}$$

During the iterative procedure, the increment of the vertical displacements of the nodes of the girder axis will decrease, so the error in the geometric approximation of the increment of the horizontal displacement will be smaller.

4. DESCRIPTION OF THE ITERATIVE PROCEDURE

In the initial step, for the cross-sectional stiffness of the rod EI and EA, the displacements $v_{(0)}$ and the ratio $(dv/dx)_{(0)}$ at the discretization points are calculated. The dependence v(x) and (dv/dx)-x is formulated using polynomials in the range 0-L [11]. Next, the change in longitudinal distance Δu of the rod nodes *i-k* is calculated. In the first iterative step of the deformation calculation, the cross-sectional forces $M_{(1)}$, $V_{(1)}$, $H_{(1)}$ determined on the deformed rod and the geometry of the systems $L_{(0)}$ i $\psi_{ik,(0)}$. are taken. In the n-th iterative step, the expression for the displacement increment is of the form:

$$\Delta v_{(n)} = \int_0^x \overline{M}_{(n)} \cdot \frac{\Delta M_{(n)}}{EI \cdot \cos\psi_{ik,(n-1)}} dx + \int_0^x \overline{N}_{(n)} \cdot \frac{\Delta N_{(n)}}{EA \cdot \cos\psi_{ik,(n-1)}} dx \tag{15}$$

The vertical displacement of the rod in the n-th iterative step is of the form:

$$v_{(n)} = v_{(n-1)} + \Delta v_{(n)} \tag{16}$$

The change in the horizontal distance of the nodal points of the rod i is determined in the form:

$$\Delta u_{(n)} = \left[\int_0^L \varepsilon_{(n-1)}\, dx - \frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2_{(n)} dx\right] \qquad \left[\left(\frac{dv}{dx}\right)^2 < 1\right]$$
$$\Delta u_{(n)} = tg\,\Delta\psi_{ik,(n)} \cdot L_{(n-1)} \cdot \sin\psi_{ik,(n-1)} \qquad \left[\left(\frac{dv}{dx}\right)^2 > 1\right] \tag{17}$$

The iteration procedure is carried out until the difference between the solutions for deformations in two iterative steps (n-1 and n) is small according to the adopted norm in the form:

$$e = v_{(n)} - v_{(n-1)} \tag{18}$$

5. NUMERICAL EXAMPLES
5.1 Example 1

Through a numerical example, the cantilever loaded with concentric force at the end of the rod is analysed. The proposed procedure for calculating the geometric nonlinearity of a rod is described through the example. $E=1,8x10^8$ kN/m^2; EI=0,1875 kNm^2; EA=90 000 kN.

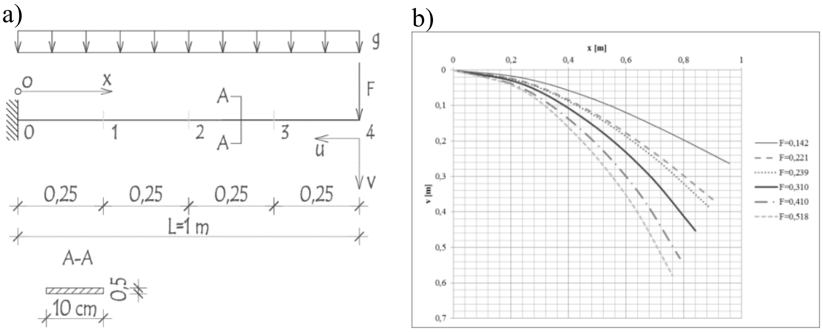

Figure 4. a) Example 1, b) rod displacement diagram

The rod is divided into four elements. At the points of discretization, the vertical and horizontal displacements of the nodes are determined in an incremental-iterative way. For the analysed force intensities F, the results of node displacement are shown in Figure 4.b and Table 1. The force-displacement dependence in Figure 5 is shown for the following calculation cases: Lin – calculation according to linear theory; Lit – calculation according to the literature [2]; Man – manual calculation (geometric nonlinearity); SCIA – SCIA program (geometric nonlinearity) [7].

Table 1. Numerical values of cantilever beam displacement

	F [kN]					
v [m]	**0,142**	**0,221**	**0,239**	**0,310**	**0,410**	**0,518**
v_0	0	0	0	0	0	0
v_1	0,024	0,033	0,035	0,041	0,052	0,056
v_2	0,084	0,119	0,124	0,147	0,177	0,194
v_3	0,168	0,235	0,246	0,288	0,341	0,374
v_4	0,263	0,370	0,390	0,453	0,531	0,579
u_5	0,042	0,092	0,108	0,160	0,214	0,239

The iterative procedure of manual calculation according to the large displacement theory for the load intensity F=0,31 kN is shown in Table 2.

Table 2. Iterative calculation procedure in increment for force F=0,31 kN

Node	M [kNm]	ΔM [kNm]	$\overline{M}_4$	$\overline{M}_3$	$\overline{M}_2$	$\overline{M}_1$	Δv [m]	v [m]	Δu [m]	Δu0-1	Δu1-2	Δu2-3	Δu3-4	x1	x2	x3	x4	tg ψik		cos ψik		sin ψik		u [m]
0	0,294	0,063	0,892	0,695	0,480	0,247	0,000	0,000										0,180		0,984		0,177		0,000
1	0,209	0,046	0,645	0,448	0,233		0,009	0,044											0,507		0,892		0,452	0,005
2	0,132	0,029	0,412	0,215			0,033	0,157	0,165	0,005	0,028	0,053	0,078	0,245	0,467	0,664	0,836	0,769		0,793		0,609		0,033
3	0,062	0,014	0,197				0,062	0,308											1,017		0,701		0,713	0,086
4	0,000	0,000					0,093	0,483																0,164
2. iteration																								
0	0,276	-0,018	0,836	0,664	0,467	0,245	0,000	0,000										0,167		0,986		0,165		0,000
1	0,192	-0,018	0,591	0,419	0,222		-0,003	0,041											0,465		0,907		0,422	0,005
2	0,118	-0,014	0,369	0,197			-0,011	0,146	0,159	0,005	0,025	0,050	0,079	0,245	0,470	0,670	0,841	0,702		0,818		0,575		0,030
3	0,054	-0,008	0,172				-0,022	0,286											0,959		0,722		0,692	0,080
4	0,000	0,000					-0,033	0,450																0,159
3. iteration																								
0	0,277	0,002	0,841	0,670	0,470	0,245	0,000	0,000										0,167		0,986		0,165		0,000
1	0,194	0,002	0,596	0,425	0,225		0,000	0,041											0,469		0,905		0,425	0,005
2	0,119	0,001	0,371	0,200			0,001	0,147	0,161	0,005	0,025	0,050	0,080	0,245	0,470	0,670	0,840	0,707		0,817		0,577		0,030
3	0,054	0,000	0,171				0,002	0,288											0,971		0,718		0,696	0,080
4	0,000	0,000					0,003	0,453																0,160

The force-displacement ratio of the rod in the analysis of geometric nonlinearity are nonlinear [9], Figure 5.a. Nonlinear relations of forces and displacements are a consequence of changes in the stiffness of the system. The variation in system stiffness occurs due to a change in geometry.

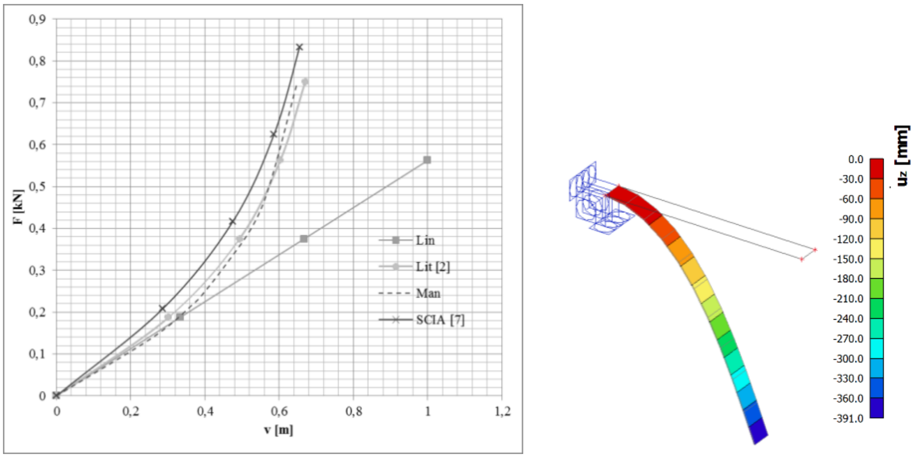

Figure 5. a) F-v ratio, b) rod displacement by F=0,31 kN in SCIA program [7]

5.2 Example 2

In the example, the static-deformation state of a simple beam loaded with concentric forces F and H is analyzed. The stress-strain relationship of the material is linear. The load on the beam is applied in increments, and then in each increment the solution of displacement and shear forces was determined iteratively. b/h=10/0,5 cm; E=2x10^8 kN/m^2;EI=0,2083 kNm^2; EA=100000 kN.

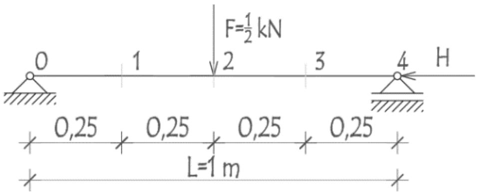

Figure 6. Example 2

The vertical and horizontal displacements at variable force H were determined in the nodes. The relations between the finite deformations and the force H are shown in Figure 7.

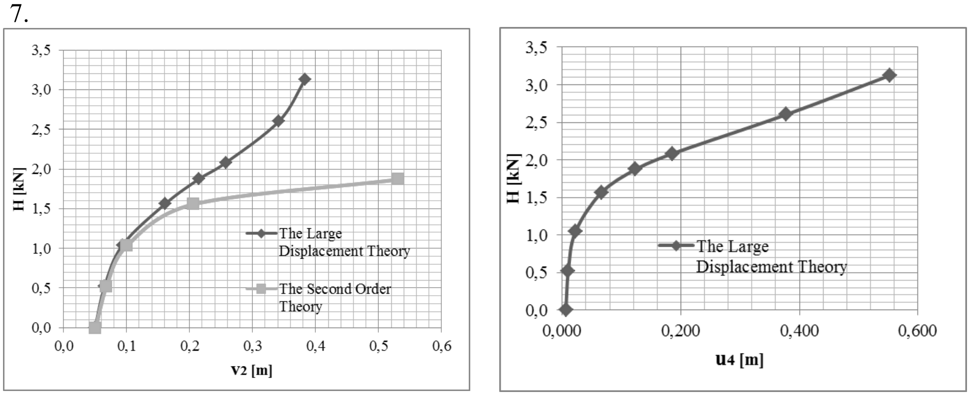

Figure 7. a) Relation H-v2, b) Relation H-u4

Figure 8. Shows the deformation line in the final position of the rod [7].

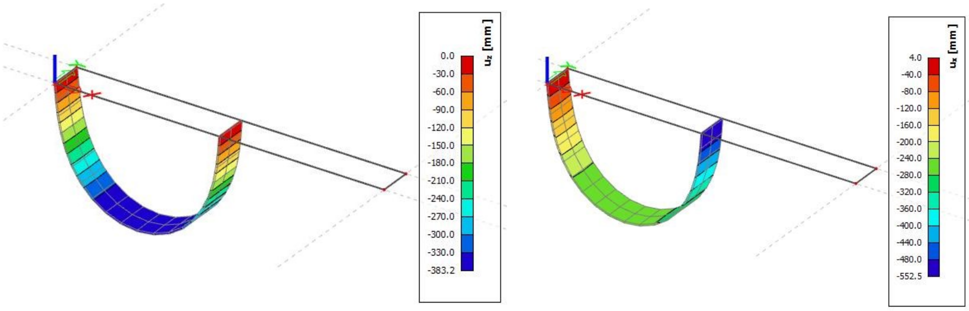

Figure 8 . a) Vertical displacements, b) horizontal displacements for $H=15EI/L^2$

The calculation of forces and deformations of a simple beam according to the second-order theory can be performed up to the critical force $P_{kr}=\pi^2 EI/L^2=2{,}06$ kN [5]. According to the large displacement theory, the calculation can be performed for all load values. The results of the calculation of shear forces and deformations according to the large displacement theory and the second-order theory of the are shown in Table 3.

Table 3. Displacement and bending moments of a simple beam

			The Large Displacement Theory			The Second Order Theory		
	H(EI)	H [kN]	v_2 [m]	u_4 [m]	M_2 [kNm]	v_2 [m]	u_4 [m]	M_2 [kNm]
H<Pkr	0	0,000	0,050	0,006	0,125	0,050	0,00000	0,125
	$2{,}5EI/L^2$	0,521	0,065	0,010	0,156	0,067	0,00001	0,160
	$5EI/L^2$	1,042	0,094	0,022	0,216	0,100	0,00001	0,230
	$7{,}5EI/L^2$	1,562	0,162	0,066	0,365	0,205	0,00002	0,440
	$9EI/L^2$	1,875	0,215	0,124	0,504	0,531	0,00002	1,116
H>Pkr	$10EI/L^2$	2,083	0,258	0,186	0,629			
	$12{,}5EI/L^2$	2,604	0,342	0,378	0,953			
	$15EI/L^2$	3,125	0,383	0,553	1,230			

6. CONCLUSION

The analysis of the forces equilibrium on the deformed beam gives more realistic results of the calculation of forces and deformations. The large displacements of the system as a geometrically nonlinear problem are determined by the introduction of higher order elements into the geometric connections of the rod element. The balance of forces is established in the final position of the rod. The solution of nonlinear equations is determined by the incremental-iterative procedure. From the presented examples we conclude that the results of the calculation of forces and deformations according to the large displacement theory differ significantly from the linear theory and the second-order theory. In the example of a rod with a clamp at the end due to a load, in addition to vertical displacements, horizontal displacements of the rod occur. By establishing a balance of forces on a deformed system, a redistribution of static influences occurs. The vertical displacements obtained according to the large displacement theory on a simple beam in Example 2 are smaller than the calculation results according to the second-order theory. The reason for this is the lower bending moment due to the shortening of the simple beam. According to the analysed examples of calculations using the first-order theory, second-order theory, and the large displacement theory, we conclude that the results of the calculation differ significantly depending on the beam. At small rod displacements, second-order theory gives satisfactory calculation results. In cases of large displacements and forces $H > P_{kr}$ of the rod, it is necessary to apply the calculation according to the large displacements theory.

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НУМЕРИЧКА АНАЛИЗА ШТАПА ПРЕМА ТЕОРИЈИ КОНАЧНИХ ДЕФОРМАЦИЈА

Резиме: *У раду је приказан поступак нумеричког моделирања геометријске нелинеарности штапа. Прорачун пресјечних сила, помјерања и обртања чворова је урађен итеративним методама на деформисаном систему. Описаним поступком равнотежно стање се успоставља у коначном положају штапа. У процесу* *деформације јавља се прираст пресјечних сила и деформација штапа. Приказаним методама прорачуна моделира се геометријска нелинеарност са константном и промјењивом крутости попречног пресјека штапа. Прорачуни су урађени нумерички, а резултати контролисани примјеном програмског пакета СЦИА. Кроз нумеричке примјере је презентиран поступак прорачуна и извршена анализа резултата.*

Кључне ријечи:*Теорија коначних деформација, геометријска нелинеарност штапа*