GENERALIZATION OF STRESS–DEFORMATION RELATIONS FOR CONCRETE

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Summary:*The paper aims is to compare several different stress-deformation relations by integral equations of concrete as form of its general solution.With general solution of many tasks of prestressed and composite concrete structures is easily to find simple and explicite results.Here are shown possibility to have properly used theory of aging or heredity and heredity theory of aging for the* <u>*working loads*</u>*. Very usefully is to have partition factors for total creep as solution for all considered relations.*

Key words: *Concrete,mathematical model, creep (viscosity), Maxwell's model.*

1. INTRODUCTION

The paper aims to compare several different σ-ε concrete relations as form of its general solution. In previous studies, based on the consideration of the rheological properties of concrete as a viscoelastic material, a rheological model was required which would correspond to Dishinger's differential equation [9]. In the author's paper, it was concluded that it is equivalent to the differential equation of the Maxwell model with variable mechanical characteristics in Hook's and Newton's body.
It should be shown here that almost all rheological relations for concrete can be obtained from theirs general mathematical model. Adopting the general form of relations solutions across different theories are considered as special cases of general relation.
Also, the possibility will be given in more detail that, based on measured data for the total values of the creep function in laboratories, during the construction of significant structures, or data that can be found in national regulations for certain climatic regions, it can decide which concrete theory should be applied in the calculation of the deformations due to the constant or monotonically variable load of concrete structures.

2. INTEGRAL EQUATIONS AS A MODEL

Studies of these relationships begin with Boltzman's observations (1877), then they continue with the definite integral Volterr equation (1931), which today carries its name. Major new proposals considerations start from this equations. The same approach will be

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adopted here.The relation between the stress and the deformationn of concrete, as an integral equation, is written in the following two shapes:

$$E_0\varepsilon(t)=\sigma(t)+\int_{\tau_0}^{t}\sigma(\tau)K(t,\tau)d\tau \qquad (1)$$

$$E_0\varepsilon(\theta)=\sigma(\theta)+\int_{\theta_0}^{\theta}\sigma(\theta_\tau)K(\theta-\theta_\tau)\,d\theta_\tau \qquad (2)$$

The first form is for real time t and the other for the replacement time θ(or 'Pseudotime' [ ]) ,. which are introduced because then are easier formation of explicit solutions for stresses and deformations in cross sections of concrete structures.
The kernel K (t, t) is determined in the general form based on the experience of using the Dish ysinger equation in the theory of aging and the application of a viscoelastic rheological model to the theory of inheritance (Ržanicin, Arutjunjan, Courbon, Maslov, etc.).
The general type of the kernnel of the integral equation of form (1) and form (2) with a linear combination of a finite number of exponential functions will be adopted:

$$K(t,\tau)=\sum_{i=1}^{m}c_i\cdot e^{-a_i(\theta_t-\theta_\tau)}\cdot\frac{d\theta_\tau}{d\tau} \qquad (1a)$$

$$K_1(\theta-\theta_\tau)=\sum_{i=1}^{m}c_i\cdot e^{-a_i(\theta-\theta_\tau)}\cdot d\theta_\tau \qquad (2a)$$

where the c_i and a_i are the parameters of the karnels, and $\theta_t=\theta(t)$a function expressing the influence of the variability of the concrete characteristics, during the time of the aging of the concrete .
The shape of the kernel under (1a) is with real time, and the kernel under (2a) holds for the equation (2a) with replacing time.
The inheritance as a characteristic of young concrete is expressed by the difference of functions (θ_t - θ_τ). Instead of the real time $\theta_\tau=\tau$ in expression (1a), the replacement time is introduced $\theta_\tau=\varphi(\tau)$ in expression (2a), which denotes the viscous creep function of the concrete.
The solution of the equation (1) with the kernel (1a), and the equation (2) with the kernel (2a) is now:

$$\sigma(t)=E_0\,\varepsilon(t)-E_0\int_{\tau_0}^{t}\varepsilon(\tau)R(t,\tau)d\tau \qquad (1b)$$

$$\sigma(\theta)=E_0\,\varepsilon(\theta)-E_0\int_{\theta_0}^{\theta}\varepsilon(\theta_\tau)\,R(\theta-\theta_\tau)\,d\theta_\tau \qquad (2b)$$

where the rezolventas are:

$$R(t,\tau) = \sum_{i=1}^{m} d_i \cdot e^{-b_i(\theta_t - \theta_\tau)} \cdot \frac{d\theta_\tau}{d\tau} \tag{1c}$$

$$R_1(\theta - \theta_\tau) = \sum_{i=1}^{m} d_i \cdot e^{-b_i(\theta - \theta_\tau)} \cdot d\theta_\tau \tag{2c}$$

In terms of (1c) and (2c) the parameters of the rezolutions are di and bi, which can be determined by Laplase transformations. <u>In the case m = 1, which is from the engineering point of view the most important in practice, d1 = c1 = c and b1 = b = a + c. [4].</u> [v. 8].

The original form of the term was intended only for the tasks of the theory of heredity with replacement $\theta zam = t - \tau$ [see 4], which now, as is seen, replaced $\theta zam = \theta t - \theta\tau$ The original form of the term was intended only for the tasks of the theory of heredity. This replacement was, also, intended in postgraduate work for theory of aging (1974) [8].

Both kernel relatios are designed to be easier work, with time-substitution or with real-time, in order to extend the field of application to multiple directions of aging theory (see Table 1).

Due to wider representations of the theory of heredity.in practice, with real time for m = 1 in (1a), but also for those properties of concrete that depend on the number of models related to the series (m> 1), the relations with the sum of several members-models are shown and for relations (2a) with replaceable time.Calculation procedures for m> 1 are considerably more complex and contribute little to its accuracy.

From the general integral equation (1) with the kernel (1a) or the equation (2) and the kernel (2a), the following stress-deformation relations are formed:

I. <u>Theory of Aging (Dischinger [6])</u>

If in the general expression of the kernel with one member (2a), for the case m = 1, the constants are added according to <u>Table 1</u> in the next page, from the general solution for the rezolution (2c), the b and d constants are obtained according to the theory of aging, which are also in it shown.

The general relationship confirmed the accuracy when comparing the obtained values with data from rheological models [15]. The table presents the first solution of integral equations in the function of the replacement time for Theory I to III, and for theory IV with real time.

Table 1. Expressions for kernel and rezolution of integral equations

	θ	E	a	b	c	$K_1(\varphi,\varphi_\tau)$	$R_1(\varphi,\varphi_\tau)$
I	φ	E_0	0	1	1	1.0	$e^{-(\varphi-\varphi_\tau)}$
II	φ	$\frac{E_0}{1+\bar{\varphi}}$	0	$\frac{1}{1+\bar{\varphi}}$	$\frac{1}{1+\bar{\varphi}}$	$\frac{1}{1+\bar{\varphi}}$	$(1/(1+\bar{\varphi}))\, e^{-(1/(1+\bar{\varphi}))(\varphi-\varphi_\tau)}$
III	φ	E_0	1	$1+r$	r	$r\cdot e^{-(\varphi-\varphi_\tau)}$	$r\cdot e^{-(1+r)(\varphi-\varphi_\tau)}$
IV	t	E_0	β_∞	$\beta_\infty(1+\varphi_\infty)$	$\varphi_\infty\beta_\infty$	$\beta_\infty\varphi_\infty\, e^{-\beta_\infty(t-\tau)}$	$\varphi_\infty\beta_\infty\, e^{-\beta_\infty(1+\varphi_\infty)(t-\tau)}$

I – Theory of aging (Dischinger)
II - Second shape Theory of aging (Rüsh)
III - Hereditary theory of aging (Ivkovich)
IV - Theory of heredity (Rzanicin, Courbon, ,and other)
For theory IV are: $K(t,\tau)$ and $R(t,\tau)$

For all theories is : $d = c$

The integral equation with the kernel (under I) in Table 1 is simply proseed, by differentiation on the Diesinger differential equation::

$$\frac{d\varepsilon}{dt} = \frac{1}{E(t)}\frac{d\sigma}{dt} + \frac{\sigma}{E_0} \qquad (3)$$

The paper [9] showed that it is equivalent to a modified Maxwell model with a changeable modulus of elasticity and a viscosity coefficient over time. Due to the importance of the Modified Maxwell Model ($\tilde{M}$) and the viscoelastic model (VE), they are given in Fig. 1, but without a more detailed description [see 15].

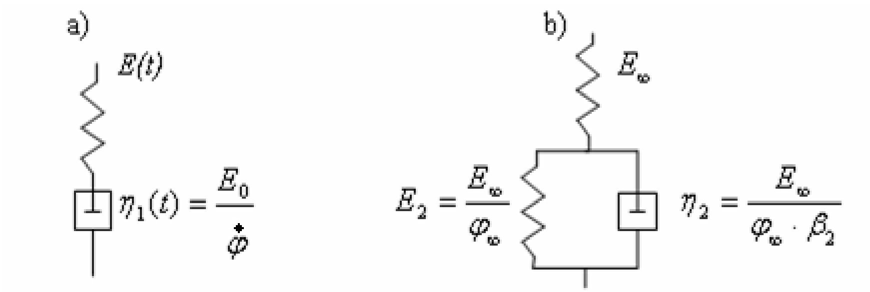

Fig 1. a) Modified Maxwell model b) Modified VE model

The following relationship was found for the coefficient of viscosity of the model ($\tilde{M}$) in the figure under (a):

$$\eta_1(t) = \frac{E_0}{\frac{d\varphi(t)}{dt}} \tag{4}$$

In aging theory, concrete can be understood as Maxwell's fluid, whose viscosity increases over $>$ time. When t ∞ concrete becomes a solid $>$ body, because then η ∞.. In the same picture (under b.), a viscoelastic model with parameters whose values are to be known and measured is shown.
Dishinger's differential equation is widely accepted by many researchers and engineers in practice for the computational analysis of concrete structures [1] [5] [14] [17].

II. <u>Modified aging theory (Rüsh [3])</u>
The procedure will be repeated as in the previous case I. Using Table 1. for the adopted Rüsh parameters, it follows from the parameters a and c, then next b and d, the kernel, and the rezolution. The general relations shows that the results are the same as those given in[10] .Then the kernel:

$$K_1(\varphi_t, \varphi_\tau) = 1/(1+\overline{\varphi})$$

is written in the function of the replacement time φ.

The rheological model is called the modified Maxwell model, in order to distinguish it from its classical form, with parameters that have a certain significance in physics [9], and where $E\varphi = E_0 / (\overline{1} + \varphi)$ is a fictitious modulus of elasticity. The relation was created in order to improve the Dishinger differential equation.

III. <u>The next theory of aging (Ivkovic [10])</u>
Applying the same procedure as in the previous cases with one member m = 1, and for the adopted parameters of Ivkovic a and c follow from Table 1. b and d, kernel and rezolution.
The general relation shows that the results are the same as those given in [10].
Then it's a kernel:

$$K_1(\varphi t, \varphi \tau) = r \cdot e^{-(\varphi t - \varphi_\tau)} \tag{6}$$

where the replacement time function.$\varphi_t = \varphi$ is given as function of time(t).
The expression $(\varphi_t - \varphi_\tau)$, as the upper index of the exponential kernel, belongs to the forms of hereditary aging theory.
The integral equation (2) is now equivalent to the displasments of the modified rheological model VE, as shown in Fig.1b, but with other model parameters. It is considered that the model can successfully describe the loading and unloading area [10].

IV. Theory of heredity

If the same procedure as in the previous cases, with one member m = 1 is applied, and we adopt the parameters a and c, from Table 1. follows the parameters b and d, the kernel and the rezolution of the integral equation.

The general relation shows that the results are the same as those given in the papers [4] [10] [17].

Now it's a kernel:

$$K(t,\tau)=\varphi_\infty \cdot \beta_\infty \cdot e^{-\beta_\infty (t-\tau)} \tag{7}$$

given as the function of real time $\theta t = t$.

The term $(t-\tau)$, as the upper index of the exponential kernel, belongs to the forms of the theory of heritage. In this form it often appears in the literature [4] [17].

The corresponding rheological model has already been described as a viscoelastic model of extended elasticity, but now with a viscosity coefficient η (t) = const [5].If we associate m Kelvin models and one Hukov body with E_0 = const, the kernel of the integral equation is as the function of real time t [4] [17] :

$$K(t,\tau)=\sum_{i=1}^{m} \varphi_{\infty,i}\, \beta_{\infty,i} \cdot e^{-\beta_{\infty,i}(t-\tau)} \tag{8}$$

Review of results for cores and resolutions of integral equations, i.e. theory from I to IV are shown in the above table.

V. The Hieditery Aging Theory (Ellston-Jordaan [13])

If we in the general kernel (1a) adopt m = 2, $\varphi = \theta\tau$, and relations parameters

$a_1 = 0$, $c_1 = 1$, $a_2 = \beta_\infty$ i $c_2 = \varphi\, \beta_\infty$ is obtained

$$K(t,\tau)=\left[1+\varphi_\infty \beta_\infty e^{-\beta_\infty (\varphi t - \varphi_\tau)}\right] \frac{d\varphi_\tau}{d\tau} \tag{9}$$

from the general solution kernel to this proposal [11].]. The kernel contains two members that are real-time functons.

The paper [10] shows the way to obtain the rezolution in the case of m> 1. It is necessary to emphasize Jordana's statement, that “the resourcing theory is not complete "in the summary of his work.

Hereditary theory has formed solutions for m> 1 in [4], but they differ in relation in the way of defining rheological parameters in our model. The expressions in Table 1 for Theory IV and expressions in the paper can be compared [4].

In summary, the following concludes:

1.Volume rheological models for concrete according to theories I to IV. are derived from the general mathematical relation defined by the integral equation (1) or (2) depending on the choice of the function $\theta(t)$, the kernel parameters and the choice of the alternating or real time observation of the viscous creep process of the concrete.

2. The solutions of the integral equation are sought only for the general case, and the relationships according to Theories I to IV are treated as special cases (m = 1);

3. For m = 1, explicit solutions of many problems of concrete structures are obtained. However, some general considerations are also shown for m> 1, because such procedures are present in many publications. [4] [17].

3. PARCIAL PARTS OF THE COEFFICIENT OF CREAP

The determination of concrete properties is successfully performed using: experiments carried out in labs, using PBAB '87 or EC2, and using data from literature [2] [15].

When data on the modulus of elasticity, coefficients of creep and shrinkage of concrete, as well as other concrete characteristics, have been adopted, they should be applied correctly in the concrete behavior during long-term loading.

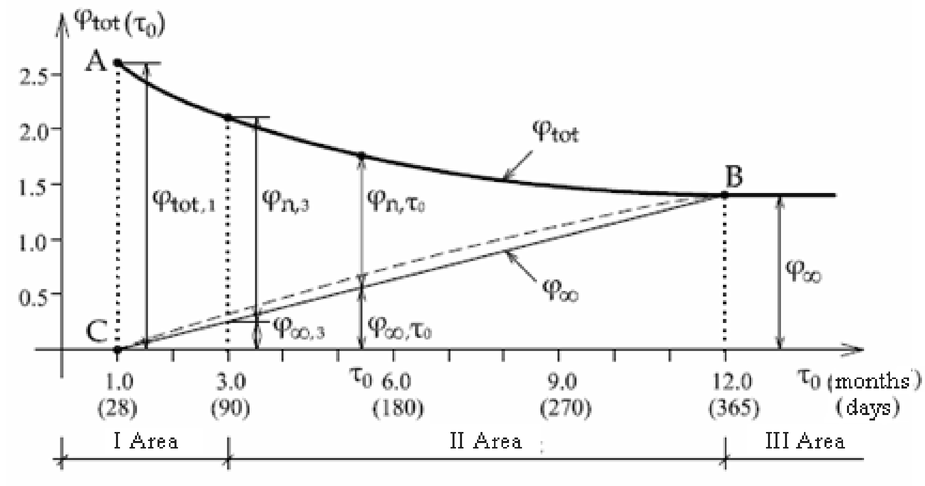

Fig. 2. Distribution of the values of the final creep factor $\varphi_{tot}(\tau_0)$ to the the theory of aging φ_{n,τ_0} and the theory of hiredity φ_{∞,τ_0} (Proposal by author)

On Fig. 2 shows the curve of the final value of the coefficient of creep of concrete from the age of $\tau_0=1$ month to age $\tau_0=12$ months (data according to PBAB'87 regulations for selected humidity of 70% and sample radius r_m = 20cm).

(1) Description of curved lines and data in Fig.2

The curved line of the final values of the coefficient of creep concrete is marked with the .φ_{tot} between the points $A(\tau_{0,1},\varphi_{tot,1})$ and $B(\tau_{0,12},\varphi_{tot,12})$. Point A is obtained for the apsis value adopted $\tau_0=1$ month, and point B for the value τ_0 = 12 months.

The curve of the final values of the coefficient of creeping old concrete is marked with φ_∞ between points $C(\tau_{0,1}, 0)$ and $B(\tau_{0,12},\varphi_{tot,12})$.

At point B, at the same time, the value of φ_{tot} in both theories is reached.

Signs in Fig. 2:

AB - The curve of the final values of the creep coefficients is marked with φ_{tot}

CB - The line of the ordinate is divided into the coefficient factor $\varphi_{n,\tau0}$ according to the theory of aging and part of the ordinate for the theory of heredity $\varphi_{\infty,\tau0}$.

τ_0 - The initial load time

$\tau_{0,1}$ -1 month (Adopted value for the start time)

$\tau_{0,12}$ -_1 month (Adopted value for the apcisa for point B)

$\varphi_{n,3}$- The value of the part of the ordinate for τ_0 = 3 months (to the theory of aging)

$\varphi_{\infty,3}$- The value of the part of the ordinate for τ_0 = 3 months (to the theory of heredity)

(2) Areas of adopted values for initial loads

Considering the rate of change in the coefficient of creep of concrete φ_{tot} in Fig.2 are observed three areas relative to their age:

I . Young concrete area (for $\tau_0 <= 3$ months)

II The area expressed with the combined age (by $3 <\tau_0 <= 12$ months)

III The area of old concrete (for $\tau_0 > 12$ months)

Fields I and II for the total values of the ordinate of the creep function φ_{tot}, depending on the starting time τ_0, have a part $\varphi_{n,\tau0}$ according to the theory of aging and a part $\varphi_{\infty,\tau0}$ according to the theory of inheritance.

These parts are separated by a CB line in the same image.

The total creep values are marked with φ_{tot}.

Based on Fig. 2, the expressions for the partial values of the coefficients of creep can be written:

$$\varphi_{\infty,\tau0} = \begin{cases} \dfrac{\varphi_\infty}{\tau_{0,12}-\tau_{0,1}}(\tau_0-\tau_{0,1}) & \text{- with line CB} \\ \approx \varphi_\infty\left[1 - e^{-\beta_\infty(\tau_{0,12}-\tau_{0,1})}\right] & \text{- with courve CB} \end{cases} \quad (10a)$$

$$\varphi_{n,\tau0} = \varphi_{tot} - \varphi_{\infty,\tau0} \quad (10b)$$

The described time interval for areas should be adopted according to the data for the particular case of the courve φ_{tot}. Instead of these expressions, the test data or references can be used.

(3) Partial factors for adopted start time τ_0

For the selected τ_0 in the area of concrete age to $\tau_{0,12}$, the partial factors are:

$$\kappa_n=\varphi_{n,\tau 0} / \varphi_{tot} \quad \text{i} \quad \kappa_\infty=\varphi_{\infty,\tau 0} / \varphi_{tot} \tag{11}$$

where the distribution factor κ_n is for the theory of aging and κ_∞ for the theory of inheritance. These factors should fulfill the condition in the picture that:

$$\kappa_n+ \kappa_\infty =1. \tag{12}$$

In summary, there will be a need for at least three groups of independent concrete samples to be tested.

In the first group of samples of young concrete (for areas I and II), due to the stresses in concrete, only total dilatations of concrete ε_{tot} can be measured over time, although they also contain part of the dilatation of the old concrete.

On the second group(II) of old concrete samples, the dilatation $\varepsilon_\infty(t_n \to \infty)$ is measured at age over 1 year. (Note for different environmental conditions this limit may have a longer real time then 1 year). The end of the second area is the point B, where is the horizontal tangent to the curve AB.

The third group of samples is exposed only to the shrinkage of concrete in ambient conditions (humidity and temperature) without load. In Fig. 2, a curve is marked with dots, which gives another possibility of dividing the area(II).

(4) Partial factors in using of concrete theories

Finally, it should be seen how for τ_0 in the defined areas to adopt the following:

I . Area. It is necessary to adopt $\kappa_n=1$ and $\kappa_\infty=0$; $\varphi_{n,\tau 0}=\varphi_{tot}$
Only the theory of aging can applies, because the error is about 5%.

II. Area. Adapt to the position of the τ_0 dividing factors by expressions (10).
Make independent calculations according to the theory of aging and inheritance theory with the final values $\varphi_{n,\tau 0}$ and $\varphi_{\infty,\tau 0}$.
The stresses and deformations for both theories need to summarized in structure problems.

III. Area. It is necessary to adopt $\kappa_n=0$ and $\kappa_\infty=1$; olso $\varphi_{\infty,\tau 0} =\varphi_\infty$.
Only the theory of heritage applies, because there exist φ_∞.

If the wrong values for the final creep coefficients of the concrete are adopted, there will not be an exact stress-deformation relations to the calculation stress and deformation of strutures. In addition, the differences in the values of φ_{tot} between theories are

sometimes over 30% . The greatest changes in stresses are obtained by the theory of aging, and at least by the theory of hereditary. The hereditary theory of aging have concrete stresses values between boundary theories.
If an appropriate division is made of the value φ_{tot} in the described way , the solution will be obtained by a sum of results of two theories. In the papers [11] [14], using the general relations, very simple expressions for the stresses and deformations in the cross sections RC, PC and composite concrete structures were performed. The general type of cross sections and three special cross section cases are considered, which is often applied in practice.

4. CONCLUSIONS

Based on the above, the following conclusions are:
1. Testing the creep and shrikage of concrete requires laboratories, well equipped with equipment for long-term measurements, under controlled temperature and humidity conditions.
2. For young concrete, in our environment up to 90 days of age, the best results give a relationn of Dischinger.
3. For old concrete, for our conditions and ages of concrete over 1.- 2.g., the relation according to the theory of heridity is successfully applied .
4. For medium-sized concrete, from 30 days to 90 days, and for a more detailed analysis of small-age concrete, the described solution with partial factors can be successfully used.
5. New analyzes can improve our conclusions, especially if the accuracy of the obtained data is assessed, based on carefully and comprehensively experiments.

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UOPŠTAVANJE VEZE NAPONA I DEFORMACIJA BETONA

Rezime: *U radu biće pokazane mogućnosti da se dođe do veze između napona i deformacija betona pomoću integralne jednačine u opštem obliku. To omogućuje njihovu primenu za vrlo raznovrsne zadatke iz oblasti AB, PB i spregnutih konstrukcija. Pokazana je mogućnost da se pravilno analiziraju podaci teorije starenja i teorije nasleđa kao graničnih teorija o viskoelstičnosti betona pri radnom opterećenju. Rešenja za naslednu teoriju starenja su dovoljno tačna pod uslovom da se koriste faktori particija za totalne funkcije tečenja betona.*

Ključne reči*: Beton, matematički model, tečenje(viskoznost), teorije betona,*