SHEAR DEFORMABLE DYNAMIC STIFFNESS ELEMENTS FOR FREE VIBRATION ANALYSIS OF RECTANGULAR ISOTROPIC MULTILAYER PLATES

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UDK: 534.11 : 624.073
DOI:10.14415/konferencijaGFS 2016.027

Summary: *In this paper, two shear deformable dynamic stiffness elements for the free vibration analysis of rectangular, transversely isotropic, single- and multi-layer plates having arbitrary boundary conditions are presented. Dynamic stifness matrices are developed for the Reddy's higher-order shear deformation theory (HSDT) and the Mindlin-Reissner's first-order shear deformation theory (FSDT). The dynamic stiffness matrices contain both the stiffness and mass properties of the plate and can be assembled similarly as in the conventional finite element method. The influence of face-to-core thickness ratio and face-to-core module ratio of sandwich plate, as well as the influence of the shear deformation on the free vibration characteristics of sandwich plates have been analysed. The results obtained by proposed HSDT and FSDT dynamic stiffness element are validated against the results obtained using the conventional finite element analysis (ABAQUS), as well as the results obtained by 4-node layered rectangular finite element. The proposed model allows accurate prediction of free vibration response of rectangular layered plate assemblies with arbitrary boundary conditions.*

Keywords: *free vibrations od layered plates, dynamic stiffness method, FSDT, HSDT*

1. INTRODUCTION

Multi-layer plates composed of several laminas of different properties are widely used in different areas of engineering. Sandwich panels are usually applied in civil engineering as components of light roofs and walls to provide thermal isolation of buildings. These elements are often placed in a dynamic loading environment, thus the adequate

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computational models capable to predict the dynamic response of such structures are required. The dynamic response is usually predicted using different plate theories, where the transverse shear effects are accounted by means of the shear correction factors, or the higher-order approximation of the displacement field. Theories that consider the multi-layer structure as a single homogeneous layer are referred as equivalent-single-layer (ESL) theories [1-3]. The comparison of different ESL plate theories is given in the Reddy's overview [4] and monographs [5, 6]. To overcome the problems that may arise due to the simplifications associated with the plate kinematics in the ESL theories, the generalized layerwise plate theory (GLPT) of Reddy [7] is used to improve the representation of the kinematics. To obtain the numerical solutions for the dynamic response of plates, finite element methods (FEM) are adopted [8-14].

In the vibration analysis, the dynamic stiffness method (DSM) [15-17] is used to obtain more accurate and reliable results in comparison with the conventional FEM. The DSM uses a unique element matrix (dynamic stiffness matrix) containing both stiffness and mass properties of the structure. The selection of the DSM for solving the free vibration problem is motivated by the fact that only one dynamic stiffness element per structural member with constant material and geometrical properties can be used to accurately represent its dynamic behavior at any frequency. Different applications of the dynamic stiffness method based on the ESL plate theories are given in [18-20]. However, the main lack of the proposed methods is the inapplicability to the plates having arbitrary combinations of boundary conditions. This has been overcome in the authors' investigations [21-23], where the dynamic stiffness matrices for a completely free rectangular isotropic plate based on the Mindlin-Reissner's first-order shear deformation theory (FSDT) and the Reddy's higher-order shear deformation theory (HSDT) were developed. These solutions are free of restrictions regarding the boundary conditions.

In this paper the dynamic stiffness matrix for a completely free rectangular multi-layer plate element based on the HSDT and FSDT is presented. Three coupled Euler-Lagrange equations of motion have been transformed into two uncoupled equations of motion using a boundary layer function [24]. The proposed method enables free transverse vibration analysis of rectangular multi-layer plates with transversely isotropic layers, having arbitrary combinations of boundary conditions. The natural frequencies obtained using different dynamic stiffness multi-layer plate elements have been validated against the solutions from the commercial software Abaqus [25] and the previously verified results [13, 14]. The influence of face-to-core thickness ratio and face-to-core module ratio of sandwich plate, as well as the influence of the shear deformation on the free vibration characteristics of sandwich plates have been discussed. A variety of new results is provided as a benchmark for future investigations.

2. FORMULATION OF THE MULTI-LAYER HSDT DYNAMIC STIFFNESS ELEMENT

The geometry of rectangular multi-layer plate composed of n isotropic layers is presented in Figure 1. The assumptions and restrictions introduced in the derivation of the model are: (1) all layers are perfectly bonded together, (2) the material of each layer is homogeneous, transversely isotropic and linearly elastic, (3) small strains and small rotations are assumed and (4) inextensibility of the transverse normal is imposed.

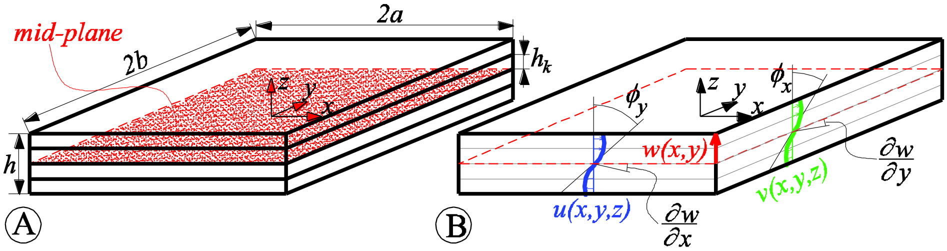

Figure 1. (A) Geometry of multi-layer plate; (B) displacement components of the HSDT

Assuming zero-deformation in the mid-plane of the plate (see Figure 1a), the displacement field of the HSDT at point (x,y,z) of a plate in the arbitrary time instant t is:

$$\begin{aligned} u(x,y,z,t) &= z\phi_y(x,y,t) - c_1 \cdot z^3\left(\phi_y(x,y,t) + \frac{\partial w(x,y,t)}{\partial x}\right) \\ v(x,y,z,t) &= -z\phi_x(x,y,t) - c_1 \cdot z^3\left(-\phi_x(x,y,t) + \frac{\partial w(x,y,t)}{\partial y}\right) \\ w(x,y,z,t) &= w(x,y,t) \end{aligned} \tag{1}$$

where ϕ_x and ϕ_y are the rotations about the x- and y-axis, respectively (Figure 1), while $c_1=4/(3h^2)$. Cross-sectional warping is accounted with a cubic approximation of the displacement field. The Euler-Lagrange equations of motion of the HSDT are derived using the Hamilton's principle [3]:

$$\begin{aligned} &-\bar{\bar{D}}_{12}\frac{\partial^2\phi_y}{\partial x\partial y} + \bar{\bar{D}}_{11}\frac{\partial^2\phi_x}{\partial y^2} + c_1\bar{F}_{12}\frac{\partial^3 w}{\partial x^2\partial y} + c_1\bar{F}_{11}\frac{\partial^3 w}{\partial y^3} - \bar{\bar{D}}_{66}\left(\frac{\partial^2\phi_y}{\partial x\partial y} - \frac{\partial^2\phi_x}{\partial x^2}\right) + \\ &+2c_1\bar{F}_{66}\frac{\partial^3 w}{\partial x^2\partial y} + \bar{\bar{A}}_{44}\left(\frac{\partial w}{\partial y} - \phi_x\right) - K_2\ddot{\phi}_x - c_1 J_4\frac{\partial \ddot{w}}{\partial y} = 0 \\ &\bar{\bar{D}}_{11}\frac{\partial^2\phi_y}{\partial x^2} - \bar{\bar{D}}_{12}\frac{\partial^2\phi_x}{\partial x\partial y} - c_1\bar{F}_{11}\frac{\partial^3 w}{\partial x^3} - c_1\bar{F}_{12}\frac{\partial^3 w}{\partial x\partial y^2} + \bar{\bar{D}}_{66}\left(\frac{\partial^2\phi_y}{\partial y^2} - \frac{\partial^2\phi_x}{\partial x\partial y}\right) - \\ &-2c_1\bar{F}_{66}\frac{\partial^3 w}{\partial x\partial y^2} - \bar{\bar{A}}_{44}\left(\frac{\partial w}{\partial x} + \phi_y\right) - K_2\ddot{\phi}_y + c_1 J_4\frac{\partial \ddot{w}}{\partial x} = 0 \\ &\bar{\bar{A}}_{44}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial\phi_y}{\partial x} - \frac{\partial\phi_x}{\partial y}\right) + c_1\bar{F}_{11}\frac{\partial^3\phi_y}{\partial x^3} - c_1\bar{F}_{12}\frac{\partial^3\phi_x}{\partial x^2\partial y} - c_1^2 H_{11}\frac{\partial^4 w}{\partial x^4} - \\ &-2c_1^2 H_{12}\frac{\partial^4 w}{\partial x^2\partial y^2} + c_1\bar{F}_{12}\frac{\partial^3\phi_y}{\partial x\partial y^2} - c_1\bar{F}_{11}\frac{\partial^3\phi_x}{\partial y^3} - c_1^2 H_{11}\frac{\partial^4 w}{\partial y^4} - 4c_1^2 H_{66}\frac{\partial^4 w}{\partial x^2\partial y^2} + \\ &+2c_1\bar{F}_{66}\left(\frac{\partial^3\phi_y}{\partial x\partial y^2} - \frac{\partial^3\phi_x}{\partial x^2\partial y}\right) - I_0\ddot{w} + c_1^2 I_6\left(\frac{\partial^2\ddot{w}}{\partial x^2} + \frac{\partial^2\ddot{w}}{\partial y^2}\right) - c_1 J_4\left(\frac{\partial\ddot{\phi}_y}{\partial x} - \frac{\partial\ddot{\phi}_x}{\partial y}\right) = 0 \end{aligned} \tag{2}$$

The higher-order stiffness coefficients $\overline{\overline{D}}_{11}, \overline{\overline{D}}_{12}, \overline{\overline{D}}_{66}, \overline{F}_{11}, \overline{F}_{12}, \overline{F}_{66}, \overline{\overline{A}}_{44}$ and mass moments of inertia K_2, J_4 are calculated by the integration of the plane stress stiffness coefficients through the plate thickness, while the above dots denote the differentiation in time. The natural (Neumann) boundary conditions of the HSDT theory are:

$$\begin{aligned}
&\delta\phi_x: \quad -\overline{M}_{xy}n_x - \overline{M}_y n_y = \overline{M}_s^*\\
&\delta\phi_y: \quad \overline{M}_x n_x + \overline{M}_{xy} n_y = \overline{M}_n^*\\
&\delta w\ :\ c_1\left[\begin{array}{l}\left(\dfrac{\partial P_x}{\partial x} + 2\dfrac{\partial P_{xy}}{\partial y} - J_4\ddot{\phi}_y + c_1 I_6\dfrac{\partial \ddot{w}}{\partial x}\right)n_x + \\ + \left(2\dfrac{\partial P_{xy}}{\partial x} + \dfrac{\partial P_y}{\partial y} + J_4\ddot{\phi}_x + c_1 I_6\dfrac{\partial \ddot{w}}{\partial y}\right)n_y\end{array}\right] + \overline{Q}_x n_x + \overline{Q}_y n_y = \overline{V}_n^*\\
&\frac{\partial w}{\partial x}: \quad -c_1\left(P_x n_x + P_{xy} n_y\right) = P_x^*\\
&\frac{\partial w}{\partial y}: \quad -c_1\left(P_{xy} n_x + P_y n_y\right) = P_y^*
\end{aligned} \tag{3}$$

The system of three coupled partial differential equations of motion (Eq. (2)) can be split into two uncoupled equations introducing the boundary layer function [24]:

$$\psi = \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} \tag{4}$$

$$\begin{aligned}
&\overline{\overline{D}}_{66}\nabla\psi - \overline{\overline{A}}_{44}\psi = K_2\ddot{\psi}\\
&C_1\cdot\nabla\nabla\nabla w + C_2\cdot\nabla\nabla w = C_3\cdot\nabla\nabla\ddot{w} + C_4\cdot\nabla\ddddot{w} - C_5\cdot\nabla\ddot{w} + C_6\cdot\ddddot{w} + C_7\cdot\ddot{w}
\end{aligned} \tag{5}$$

In Eq. (5), $\nabla = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ denotes the Laplace operator, while constants C_i are:

$$\begin{aligned}
&C_1 = \frac{c_1^2\left(\overline{\overline{D}}_{11}H_{11} - \overline{F}_{11}^2\right)}{\overline{\overline{A}}_{44}}, \quad C_2 = -D_{11}, \quad C_3 = \frac{c_1^2\left(\overline{\overline{D}}_{11}I_6 + K_2H_{11} - 2J_4\overline{F}_{11}\right)}{\overline{\overline{A}}_{44}},\\
&C_4 = \frac{c_1^2\left(J_4^2 - K_2I_6\right)}{\overline{\overline{A}}_{44}}, \quad C_5 = I_2 + \frac{\overline{\overline{D}}_{11}I_0}{\overline{\overline{A}}_{44}}, \quad C_6 = \frac{K_2I_0}{\overline{\overline{A}}_{44}}, \quad C_7 = I_0
\end{aligned} \tag{6}$$

Introducing a harmonic representation of the transverse displacement and boundary layer function, the Fourier transform of Eq. (5) can be expressed as a function of the amplitudes of transverse displacement ($\hat{w}(x,y,\omega)$) and boundary layer function ($\hat{\psi}(x,y,\omega)$), in the frequency domain (according to the procedure from [22, 23]). The amplitudes of rotations $\hat{\phi}_x(x,y,\omega)$ and $\hat{\phi}_y(x,y,\omega)$ can be expressed in terms of $\hat{w}(x,y,\omega)$ and $\hat{\psi}(x,y,\omega)$ as follows (ω is the angular frequency):

$$d_1\hat{\phi}_x + d_2\nabla\hat{\phi}_x = \overline{\overline{D}}_{66}\frac{\partial}{\partial x}\left(d_6\hat{\psi} + d_7\nabla\hat{\psi}\right) + \frac{\partial}{\partial y}\left(d_5\hat{w} + d_3\nabla\nabla\hat{w} + d_4\nabla\hat{w}\right)$$
$$d_1\hat{\phi}_y + d_2\nabla\hat{\phi}_y = \overline{\overline{D}}_{66}\frac{\partial}{\partial y}\left(d_6\hat{\psi} + d_7\nabla\hat{\psi}\right) - \frac{\partial}{\partial x}\left(d_5\hat{w} + d_3\nabla\nabla\hat{w} + d_4\nabla\hat{w}\right) \tag{7}$$

where the constants d_i are:

$$d_1 = \overline{\overline{A}}_{44} - \omega^2 K_2 - c_1 J_4 - \omega^4\frac{c_1 J_4 K_2}{\overline{\overline{A}}_{44}},\ d_2 = c_1\left(\overline{F}_{11} - \omega^2\frac{\overline{F}_{11}K_2}{\overline{\overline{A}}_{44}}\right),$$
$$d_3 = c_1^2\frac{\overline{F}_{11}^2 - \overline{\overline{D}}_{11}H_{11}}{\overline{\overline{A}}_{44}},\ d_4 = 2c_1\overline{F}_{11} + \overline{\overline{D}}_{11} - \omega^2 c_1^2\frac{\overline{\overline{D}}_{11}I_6 - 2J_4\overline{F}_{11}}{\overline{\overline{A}}_{44}}, \tag{8}$$
$$d_5 = \overline{\overline{A}}_{44} + \omega^2\left(2c_1 J_4 + \frac{\overline{\overline{D}}_{11}I_0}{\overline{\overline{A}}_{44}} + \omega^2\frac{c_1^2 J_4^2}{\overline{\overline{A}}_{44}}\right),\ d_6 = 1 + \omega^2\frac{c_1 J_4}{\overline{\overline{A}}_{44}},\ d_7 = \frac{c_1\overline{F}_{11}}{\overline{\overline{A}}_{44}}$$

The displacement field of the FSDT can be easely derived from Eq. (1) by setting the constant c_1 to zero, making the reduction from the HSDT to the FSDT very convenient. This reduction is not discussed here.

3. SOLUTION PROCEDURE

The amplitudes of the transverse displacement, the boundary layer function, as well as the rotations of a rectangular plate element can be presented as a sum of four symmetry contributions: symmetric-symmetric (SS), symmetric - anti-symmetric (SA), anti-symmetric - symmetric (AS) and anti-symmetric - anti-symmetric (AA) [26]. Following the procedure given in [22, 23], the deflections $\hat{w}(x,y,\omega)$, the rotations $\hat{\phi}_y(x,y,\omega)$ and $\hat{\phi}_x(x,y,\omega)$, the forces and moments in all symmetry contributions can be obtained. Then, the corresponding displacement and the force vectors ($\hat{\mathbf{q}}$ and $\hat{\mathbf{Q}}$) that contain displacements and forces on the boundaries $x=a$ and $y=b$ are obtained for all symmetry contributions. Using the Projection method [27, 28] as shown in [21-23], new vectors $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{Q}}$ are introduced, whose components are the coefficients in the Fourier series expansion of the displacements and forces on the boundaries $x=a$ and $y=b$. The relation between the force vector $\tilde{\mathbf{Q}}$ and the displacement vector $\tilde{\mathbf{q}}$ for each symmetry contribution is given as:

$$\tilde{\mathbf{Q}}_{IJ} = \tilde{\mathbf{K}}_D^{IJ}\tilde{\mathbf{q}}_{IJ}, \qquad I,J = S,A \tag{9}$$

where $\tilde{\mathbf{K}}_D^{IJ}$ is the dynamic stiffness matrix for considered symmetry contribution. The details regarding the SS case are givein in [22, 23] . Based on the procedure presented in Refs. [15, 19, 21-23], the dynamic stiffness matrix for completely free HSDT plate

element is obtained by using the following expression: $\tilde{\mathbf{K}}_D^G = \frac{1}{2}\mathbf{T}^T\tilde{\mathbf{K}}_o\mathbf{T}$, where $\mathbf{T}$ is the transformation matrix and $\tilde{\mathbf{K}}_o$ is the dynamic stiffness matrix obtained collecting the dynamic stiffness matrices of the four symmetry contributions [23]. The transformation matrix $\mathbf{T}$ relates the displacement vector $\tilde{\mathbf{q}}_o$ (containing the displacement vectors of all symmetry contribution) and the displacement vector $\tilde{\mathbf{q}}$ (containing the displacements and rotations along the boundary lines for completely free rectangular plate [23]). The transformation matrix is given in [23]. The size of the dynamic stiffness matrix $\tilde{\mathbf{K}}_D^G$ depends on the number of terms in the general solution M and is equal to $32M+12$.
The dynamic stiffness matrices of individual plates are assembled to compute the global dynamic stiffness matrix of plate assembly consisting of several plates. The assembly procedure is carried out in the same manner as in the FEM, except the plates are connected along boundary lines instead at nodes. The procedure was demonstrated in the previous works of Kolarević et al [22, 23]. The boundary conditions are applied to the global dynamic stiffness matrix by removing the rows and columns corresponding to the components of constrained displacement projections. The boundary conditions used in the numerical verification of the model are:

- Simply supported (S): $w = 0$ and $\phi_x = 0$ for the edge parallel to the y-axis and $w = 0$ and $\phi_y = 0$ for the edge parallel to the x-axis;
- Clamped (C): $w = \phi_x = \phi_y = w_{,x} = 0$ for the edge parallel to the y-axis, and $w = \phi_y = \phi_x = w_{,y} = 0$ for the edge parallel to the x-axis;
- Free (F): all displacements (w, ϕ_x, ϕ_y, $w_{,x}$ and $w_{,y}$) are $\neq 0$.

The proposed shear deformable dynamic stiffness elements have been implemented in the original program coded in MATLAB [29] and used for the numerical validation.

4. NUMERICAL VALIDATION AND DISCUSSION

The applicability of the proposed model is illustrated considering square sandwich (3-layer) panels, having the dimensions $2a\times2b = 2.0\times2.0m$ and the total thickness $h = 0.2m$. The face thicknesses are $t_f = 2mm$ ($h/t_f = 100$). The panels are clamped along all edges and composed of two rigid isotropic faces ($E_f = 100GPa$) and core having the Young's modules varying from 0.2-100GPa (where $E_f/E_c = 1$ corresponds to the isotropic plate). The Poisson's ratio and the mass density of both faces and core are constant: $\nu_f = \nu_c = 0.3$ and $\rho_f = \rho_c = 3000kg/m^3$. The plates are analized using four different numerical models: FSDT dynamic stiffness element - FSDT DSM (shear correction factor k=5/6 and 2 elements), HSDT dynamic stiffness element - HSDT DSM (2 elements), 4-node GLPT layered rectangular finite element with reduced integration - GLPT P4R (20×20 elements) (see [21, 22]) and 4-node conventional shell element with reduced integration (S4R), built in the commercial software Abaqus (100×100 elements). In the calculations performed by the dynamic stiffness method, $M = 9$ terms in the series expansion were used to obtain the accurate solution, according to the convergence studies presented in [22, 23]. The first four natural frequencies are illustrated in Figure 2.

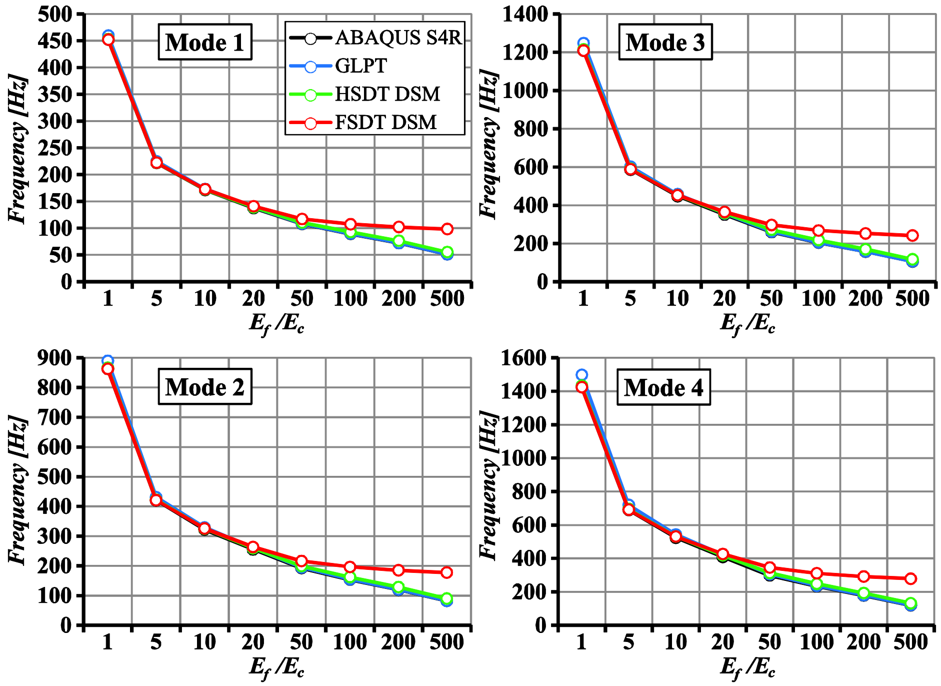

Figure 2. Natural frequencies of sandwich panels with variable E_f/E_c ratios (h/t_f = 100)

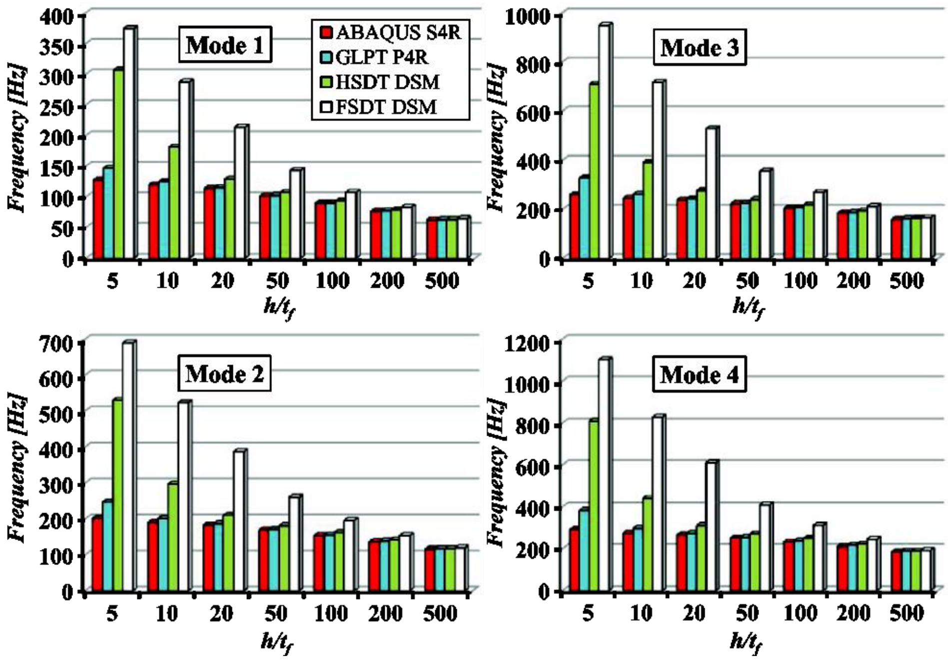

Figure 3. Natural frequencies of sandwich panels with variable h/t_f ratios (E_f/E_c = 100)

In the second part of this benchmark example, the influence of the face thickness on natural frequencies of sandwich panels isanalysed. The Young's modules of faces and core are fixed: E_f = 100*GPa*, E_c = 1*GPa*, while the ratios h/t_f are varying: h/t_f = {5, 10, 20, 50, 100, 200 and 500}. The results are presented in Figure 3.

5. CONCLUSIONS

The development of the dynamic stiffness matrix for a completely free rectangular multi-layer plate element based on the HSDT has been presented in this study, implemented in a MATLAB computer code and applied in the free vibration analysis of sandwich panels. The numerical study presented in this paper proves the ability of the proposed HSDT-based model to accurately predict the dynamic behavior of sandwich panels, with some restrictions regarding the h/t_f and E_f/E_c ratios. For h/t_f =100, the model accurately predicts the fundamental frequencies for all considered E_f/E_c ratios, varying from the isotropic plate (E_f/E_c=1) to typical sandwich panel (E_f/E_c=500). The discrepancy in the results is detected when the quality of the core layer decreases (E_f/E_c>20). For all considered cases, the results obtained using the GLPT P4R layered finite elements are in excellent agreement with the finite element solution from Abaqus. Generally, better agreement is obtained for lower modes of vibration. The FSDT dynamic stiffness element exhibits higher stiffness in comparison with other models due to the simplifications regarding the transverse shear deformation. For E_f/E_c =100, the proposed model accurately predicts the fundamental frequencies if h/t_f >50.

ACKNOWLEDGMENTS

The financial support of the Government of the Republic of Serbia - Ministry of Education, Science and Technological Development, under the Projects TR-36046 and TR-36048, is acknowledged.

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СЛОБОДНЕ ВИБРАЦИЈЕ ПРАВОУГАОНИХ ИЗОТРОПНИХ ВИШЕСЛОЈНИХ ПЛОЧА ПРИМЕНОМ МЕТОДЕ ДИНАМИЧКЕ КРУТОСТИ

Резиме: *У овом раду приказане су динамичке матрице крутости за правоугаону, (трансверзално) изотропну, једнослојну и вишеслојну плочу са произвољним граничним условима, које су примењене у анализи слободних вибрација. Динамичке матрице крутости изведене су за Reddy-еву смичућу теорију плоча вишег реда (HSDT), као и за Mindlin-ову теорију плоча (FSDT). Динамичке матрице крутости садрже параметре крутости и масе разматраних плоча и могу се сабирати на сличан начин као у Методи коначних елемената (МКЕ). Разматран је утицај односа дебљине површинског слоја и језгра, као и утицај односа модула еластичности површинског слоја и језгра код сендвич плоча, као и утицај деформације смицања на слободне вибрације сендвич плоча. Резултати добијени применом динамичких матрица крутости HSDT и FSDT елемената су упоређени са резултатима комерцијалног програмског пакета Abaqus и резултатима заснованим на слојевитом правоугаоном коначном елементу са 4-чвора. Предложени модели омогућавају прецизно одређивање динамичког одговора система правоугаоних плоча са произволјним граничним условима.*

Кључне речи: *слободне вибрације слојевитих плоча, метода динамичке крутости, Mindlin-ова теорија плоча, смичућа теорија плоча вишег реда*